Second Memoir
Studies on the Primitive Parallelohedra
Second Part
Domains of Quadratic Forms

Georges Fedosevich Voronoi

Monograph Translations Series

Series Editor: Vaen Sryayudhya

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God's Ayudhya's Defence
Bangkok

11th February, 2007

Catalogue in Publication Data
G. F. Voronoi
Second Memoir, Studies on the Primitive Parallelohedra, Second Part, Domains of Quadratic Forms:
– Bangkok, Kittix, GAD, 2007
132 p.
1. Second Memoir, Studies on the Primitive Parallelohedra, Second Part, Domains of Quadratic Forms. I. Voronoi, G. F. II. Tyabandha, Kit III. Mathematics.
510
ISBN 978–974–88249–1–8

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Published by
God's Ȧyudhyạ's Defence
A Non-profit School of
Mathematics, Language and Philosophy
1564/11 Prajarastrasaï 1 Road
Bangzue, Bangkok 10800, Thailand

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Typeset using TEX

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In Thailand, Baht 1,200
Elsewhere consult distributor's quotation

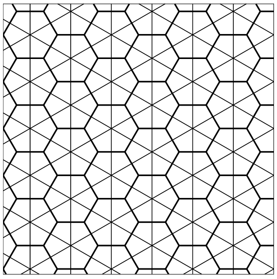
Distribution
Internet-searchable world-wide

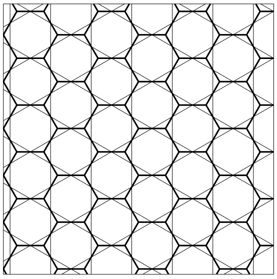
To God

Preface

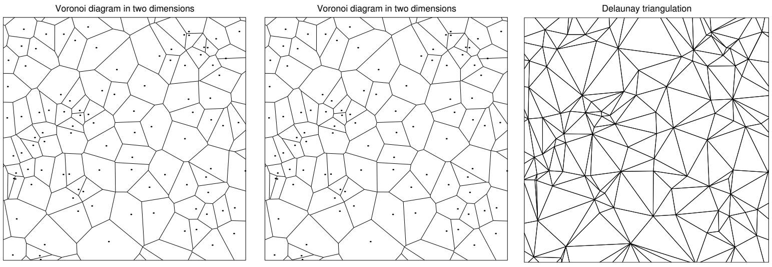
Tessellation or tiling is an arrangement of domains which fills the space. In two dimensions these domains are polygons and in three dimensions they are polyhedra. There are only three regular polygons that can tile. These are the equilateral triangle, the square and the regular hexagon, and they make up what is called *regular* tessellations. There are eight *semi-regular* tessellations. The three regular ones comprise two pairs of dual lattices. The square is dual to itself. The regular hexagon and the equilateral triangle are dual to each other.

Apart from the dual lattice, one has a covering lattice which is defined to have vertices at the mid-points of edges in the original lattice and to have an edge wherever there used to be a vertex joining two edges. In the following two pictures the one on the left-hand side is hexagonal lattice and triangular lattice, which are dual to each other. The one on the right-hand side is hexagonal lattice and its covering lattice called *Kagome* which comes from Japanese and which means a basket pattern. The word is pronounced differently from usual English pronunciation and is therefore written with an adopted *accent aigu* (acute accent) from French as *Kagomé*, perhaps because it is easier to do so than to explain to the reader. This is how the word literally became frenchised.

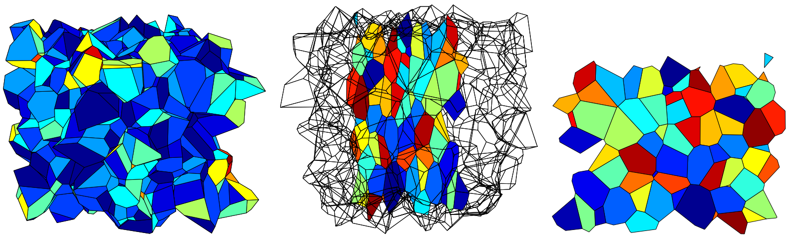

Dual lattices, hexagonal and triangular


Hexagonal and its covering lattice, Kagome

The following pictures show the Voronoi and its dual the Delaunay tessellations, as well as their superimposition.

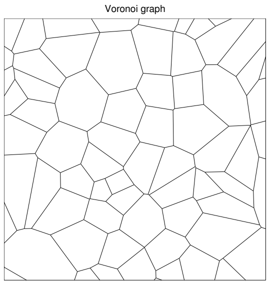

Voronoi diagram in two dimensions
Voronoi diagram in two dimensions
Delaunay triangulation

In the following pictures, (a) shows a three-dimensional Voronoi structure viewed as a solid block, (b) the same structure rotated and with a cross section running through it and (c) the cross section as viewed perpendicular to the sectioning plane.

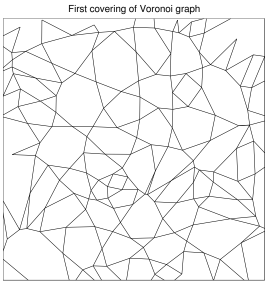

(a) (b) (c)

Let $C^n(x)$ be an n^{th}-order covering lattice of a tessellation x in two dimensions. The following pictures show a two-dimensional Voronoi tessellation together with its covering lattices C^1 up to C^3.

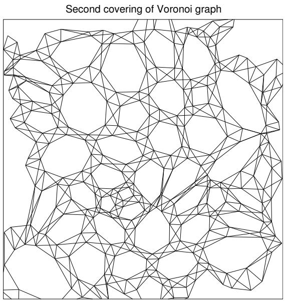
Voronoi tessellation (x)



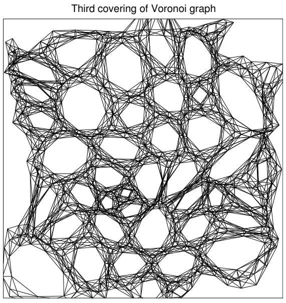
$C^1(x)$



$C^2(x)$



$C^3(x)$



The study of Voronoi tessellation has gained importance because it has been successfully used to describe many natural structures, for example it is linked to many cellular structures in nature and in Biotechnology. It has been used to study particle distributions in materials, the structure of porous media, and the structural properties of mixtures. It has been used to model anisotropic cellulose acetate and cellulose nitrate membrane, and to simulate animal cell cultures. The sand pattern formed by male mouth breeder fish, *Tilapia mossambica*, has been found to approximate a pattern of Voronoi polygons. Cluster analysis of Voronoi diagrams has been used in the analysis of clusters and scotomas in eyes in order to improve an early detection of visual field damage in glaucoma. In computer science it has been known under the name of *associative file searching* problem. Other applications are in finite element, pattern classification in statistics, dot pattern analysis, motion planning in robotics, etc. The justification for these many applications is that the statistics obtained from this set of structures compare nicely with measurements made from physical structures. Examples of such measurements are the Scanning Electro Micrograph of membranes and the pencil beam survey of the universe.

Because Voronoi structure comes up in wide and various topics, it has been known by various names. It has been called *Wigner-Seitz polygons* after Wigner and Seitz who first used it in metallurgy, *Blum's transform* after Blum who used it in visual science, *Dirichlet* and *Voronoi* tessellations after Voronoi and Dirichlet who pioneered and introduced it. At atomic level in the study of complex alloy structure it has been called *domain of an atom*. Other names include a *cell model*, a *plant polygon*, an S polygon, a *Thiessen* polygon, and *Wirkungsbereich* (sphere or radius of operation). Thiessen used it to predict rainfall average of a region in meteorology. *Plant polygon* and *area potentially available* were the terms given to area for an individual tree in a forest. The name S polygon or S mosaic sometimes arises in the field of ecology.

One way to mathematically describe a Voronoi region is

$$\Pi_i = \{x \mid d(x, x_i) < d(x, x_j)\ \forall j \neq i\}$$

where $x_1, x_2, x_3, \ldots$ are the coordinates of the nuclei with a distribution Φ of a countable set of nuclei $\{x_i\}$ in $\Re^d$ and $d(x, y)$ the Euclidean distance between x and y.

Dual to the Voronoi tessellation is the Delaunay tessellation which can be thought of as the neighbouring relationship among Voronoi regions. Delaunay (Boris Nikolaevitch Delone, 1890–1980) was the person who redefined such relationship using a concept of empty spheres which is computationally useful and gave rise to the construction method of Voronoi tessellation by mean of Delaunay triangulation.

Some more on terminology. In 1980, B. Grünbaum and G. C. Shephard called the regions *plesiohedra* to comply with a fact that they belong to a set of regions called *stereohedra* whose congruent copies fill three dimensional space without overlap except possibly at their boundaries.

New application of continuous parameters to the theory of quadratic form.

Second Memoir

Studies on the primitive parallelohedra

by Mr. Georges Voronoi in Warsaw

Second Part

Domains of quadratic forms corresponding to the various types of primitive parallelohedra

Section IV

Various types of primitive parallelohedra

[*Journal für die reine und angewandte Mathematik,*]
[V. 136, p. 67–181, 1909]

[translated by K N Tiyapan]

On the number of faces in $n-1$ dimensions of primitive parallelohedron.

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Theorem. *The number of faces in $n-1$ dimensions of a primitive parallelohedron is equal to* $2(2^n-1)$.

Let us suppose that a primitive parallelohedron R corresponding to a positive quadratic form $\sum\sum a_{ij}x_ix_j$ is defined by the independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0. \ (k=1,2,\ldots,\tau) \qquad (1)$$

We have seen in Number 48 that any system

$$\pm(l_{1k}, l_{2k}, \dots, l_{nk}),\ (k = 1, 2, \dots, \tau) \tag{2}$$

represents the minimum of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $\pm(l_{ik})$ with respect to the modulus 2. The form $\sum\sum a_{ij}x_ix_j$ possesses in this set only two minimum representations $\pm(l_{ik})$.

Let us divide the set E, composed of all the systems (x_i) of integers $x_1, x_2, \dots, x_n$, into 2^n classes

$$E_0, E_1, \dots, E_m \text{ where } m = 2^n - 1$$

with regard to the modulus 2 and suppose that the set E_0 is composed of systems the elements of which have the common divisor 2.

All the systems (2) do not belong to the different sets

$$E_1, E_2, \dots, E_m \text{ where } m = 2^n - 1.$$

It follows that

$$\tau \le 2^n - 1.$$

I argue that $\tau = 2^n - 1$. Let us suppose that among the systems (2) there are not found the systems belonging to a set E and we determine the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set E_h. Let (l_i) be a representation of this minimum.

Let us indicate by

$$(\alpha_{i1}), (\alpha_{i2}), \dots, (\alpha_{is}) \tag{3}$$

the vertices of the parallelohedron R defined by the inequalities (1) and examine the values of the function $\sum\sum a_{ij}l_il_j + 2\sum\alpha_il_i$ which correspond to the different vertices (3). Let us suppose that the sum $\sum\sum a_{ij}l_il_j + 2\sum\alpha_{ik}l_i$ be the smallest one.

By virtue of the supposition made, one will have the inequalities

$$\sum\sum a_{ij}l_il_j + 2\sum\alpha_{ir}l_i \ge \sum\sum a_{ij}l_il_j + 2\sum\alpha_{ik}l_i. \tag{4}$$

$$(r = 1, 2, \ldots, s)$$

By noticing that each point (α_i) belonging to the parallelohedron R can be determined by the equalities

$$\alpha_i = \sum_{r=1}^{s} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0, (r = 1, 2, \ldots, s)$$

one will deduce the inequalities (4) an inequality

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i \geq \sum\sum a_{ij} l_i l_j + 2\sum \alpha_{ik} l_i$$

which holds for any point (α_i) belonging to the parallelohedron R. The system (l_i) which represents the minimum of the form $\sum\sum a_{ij} x_i x_j$ in the set E_h verifies the inequality

$$\sum\sum a_{ij} x_i x_j - \sum\sum a_{ij} x_i l_j \geq 0$$

in the set E. It results that the point

$$\xi_i = -\frac{1}{2} \sum_{j=1}^{n} \alpha_{ij} l_j$$

belong to the parallelohedron R. By making in the inequality (5) $\alpha_i = \xi_i$, one notices

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_{ik} l_i \leq 0.$$

The vertex (α_{ik}) of the parallelohedron R verifies the inequality

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_{ik} l_i \geq 0,$$

therefore it is necessary that

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_{ik} l_i = 0. \qquad (6)$$

This stated, let us notice that the vertex $(\alpha_i k)$ of the primitive parallelohedron is simple.

Let us indicate by

$$\sum\sum a_{ij} l_{ir}^{(k)} l_{jr}^{(k)} + 2\sum \alpha_{ik} l_{ir}^{(k)} = 0, \quad (r = 1, 2, \ldots, n)$$

n equations which define the vertex in the parallelohedron R. As the vertex (α_{ik}) is simple, one will have an inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{ik}x_i > 0,$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, the system (0) and the systems

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}) \tag{7}$$

being excluded. By virtue of the equality (6), the system (l_i) is found among the systems (7) which all belong to the series (R).

It is therefore demonstrated that

$$\tau = 2^n - 1$$

and that the number of faces in $n-1$ dimensions of the parallelohedron R is equal to

$$2\tau = 2(2^n - 1).$$

Definition of the type of primitive parallelohedra.

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Let us examine a primitive parallelohedron R determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} \geq 0.$$

$$(k = 1, 2, \ldots, \sigma \text{ where } \sigma = 2(2^n - 1))$$

Let us indicate with

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of the parallelohedron R. One will determine with the help of equations

$$\sum\sum a_{ij}l_{ir}^{(k)}l_{jr}^{(k)} + 2\sum \alpha_{ik}l_{ir}^{(k)} = 0.$$

$$(r = 1, 2, \ldots, n, k = 1, 2, \ldots, s)$$

Each vertex (α_{ik}) $(k = 1, 2, \ldots, s)$ is characterised by n systems of integers

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}), \ (k = 1, 2, \ldots, s) \tag{1}$$

the determinant $\pm\omega_k$ of which does not cancel each other out.

Let us indicate, to make short, n systems (1) by a symbol

$$\{l_{ir}^{(k)}\}.$$

All the vertices of the primitive parallelohedron R will be characterised by a set of symbols

$$\{l_{ir}^{(1)}\}, \{l_{ir}^{(2)}\}, \ldots, \{l_{ir}^{(s)}\}. \tag{2}$$

This declared, let us examine another primitive parallelohedron R' corresponding to another positive quadratic form $\sum\sum a'_{ij}x_ix_j$. It can turn out that all the vertices of the parallelohedron R' will also be characterised by the symbols (2). One will say in this case that the two parallelohedra R and R' belong to the same type.

Definition. *One will call the various parallelohedra all the vertices of which are characterised by the set of symbols (2), "belonging to the same type."*

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One can characterise a type of primitive parallelohedra in many ways.

Let us consider a set (R) of congruent primitive parallelohedra which corresponds to a positive quadratic form $\sum\sum a_{ij}x_ix_j$.

All the vertices of parallelohedra belonging to the set (R) can be divided into classes of congruent vertices.

Let us indicate by τ the number of incongruent vertices belonging to the various classes.

Any vertex of a primitive parallelohedron is congruent to n vertices of parallelohedron, this results in that

$$S = (n+1)\tau.$$

Let (α_i) be any one vertex of parallelohedra of the set (R). One will define it with the help of $n+1$ equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A. \ (k = 0, 1, 2, \ldots, n) \qquad 3$$

The $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

characterise $n+1$ parallelohedra of the set (R) which are contiguous by the vertex (α_i). By indicating with (l_i) a system of arbitrary integers, one will characterise by the systems

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i) \qquad (4)$$

all the congruent vertices of parallelohedra of the set (R).

By attributing to the variables $l_1, l_2, \ldots, l_n$ any arbitrary values, one will characterise by $n+1$ systems (4) a class of congruent vertices.

One concludes this that a type of primitive parallelohedra can be characterised by τ systems (4).

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To have more convenience in the notations, let us introduce in our studies the linear functions

$$u = \sum_{i=1}^{n} l_i x_i, \text{ and } u_k = \sum_{i=1}^{n} l_{ik} x_i. \quad (k = 0, 1, 2, \ldots, n)$$

One will say that the symbol $(u_0, u_1, \ldots, u_n)$ characterise the vertex (α_i) determined by the equations (3);

the symbol $(u_0 + u, u_1 + u, \dots, u_n + u)$, u being a linear function in arbitrary integer coefficients, characterise a vertex congruent to the vertex (α_i).

Let us suppose that one had characterised by the symbols

$$(u_0^{(k)}, u_1^{(k)}, \dots, u_n^{(k)}), \quad (k = 1, 2, \dots, \tau) \tag{5}$$

τ congruent vertices of primitive parallelohedra belonging to the set (R). One will say that the set of symbols (5) characterise a type of primitive parallelohedra.

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Let us examine the faces in various dimensions of primitive parallelohedra belonging to the same type.

Let $P(\nu)$ be a face in ν dimensions $(\nu = 0, 1, 2, \dots, n-1)$ of parallelohedra of the set (R) defined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l_{i0} + 2\sum \alpha_i l_{i0}.$$
$$(k = 1, 2, \dots, n - \nu)$$

One will characterise this face by $n + 1 - \nu$ systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i,n-\nu})$$

or by $n + 1 - \nu$ corresponding linear functions.

$$u_0, u_1, \dots, u_{n-\nu}.$$

All the faces in ν dimensions of parallelohedra of the set (R) which are congruent to the face $P(\nu)$ will be characterised by the systems

$$(l_{i0} l_i), (l_{i1} l_i), \dots, (l_{i,n-\nu} l_i)$$

or by the corresponding linear functions

$$u_0 + u, u_1 + u, \dots, u_{n-\nu} + u.$$

By making, for example, $l_i = -l_{i0}$ one obtains $n - \nu$ systems

$$(l_{i1} - l_{i0}), (l_{i2} - l_{i0}), \ldots, (l_{i,n-\nu} - l_{i0}) \tag{6}$$

which enjoy the following property: all the determinants of the order $(n-\nu)^2$ which one can form from $n-\nu$ systems (6) do not cancel one another at the same time. Let us indicate by $\omega^{(n-\nu)}$ the greatest common divisor of these determinants. By declaring

$$x_i = \sum_{k=1}^{n-\nu} (l_{ik} - l_{i0})\xi_k, \tag{7}$$

one will present a system (x_i) of integers by the linear forms where $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ are integer or rational numbers which belong to $\omega^{(n-\nu)}$ sets

$$\xi_k = \vartheta_{kr} + y_k \ \ (k = 1, 2, \ldots, n - \nu; r = 1, 2, \ldots, \omega^{(n-\nu)}) \tag{8}$$

where $y_1, y_2, \ldots, y_{n-\nu}$ are arbitrary integers. Among the sets (8) is found a set where $\vartheta_{kr} = 0$, $k = 1, 2, \ldots, n-\nu$ and which is composed of integer values of $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$.

In the case $\omega^{(n-\nu)} = 1$, the equalities (7) are possible only on condition that the number $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ be integer.

The set (8) play an important role in the subsequent studies.

Let us indicate by the symbol $\sigma_{n-\nu}$ the number of incongruent faces in ν dimensions of primitive parallelohedra belonging to the type examined. By indicating with the symbol S_ν the number of faces in ν dimensions of corresponding primitive parallelohedron, one will have a formula

$$S_\nu = (n + 1 - \nu)\sigma_{n-\nu}. \ (\nu = 0, 1, 2, \ldots, n - 1) \tag{9}$$

Definition of the set (L) of simplexes characterising a type of primitive parallelohedra

Let us suppose that $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \tag{1}$$

characterise a vertex of primitive parallelohedra belonging to the type examined.

Definition I. *One will call correlative to the vertex of primitive parallelohedra characterised by the systems* (1) *a simplex L having $n+1$ vertices*

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

The simplex L presents a set of points determined by the equalities

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$

$$(k = 0, 1, 2, \ldots, n)$$

Let us indicate by (L) the set of simplexes correlative to the various vertices of a set (R) of primitive parallelohedra belonging to the type examined.

Definition II. *One will say that a type of primitive parallelohedra is characterised by the set* (L) *of simplexes.*

One will call congruent two simplexes characterised by the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \text{ and } (l_{i0}+l_i), (l_{i1}+l_i), \ldots, (l_{in}+l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

All the simplexes of the set (L) can be divided into classes of congruent simplexes; the number of classes is expressed by the symbol σ_n defined by the formula (9) of the previous number.

With the help of equations

$$x_i = \sum_{k=0}^{n-\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n-\nu} \text{ and } \vartheta_k \geq 0,$$

$$(k = 0, 1, 2, \ldots, n - \nu)$$

one will determine a face in $n - \nu$ dimensions of the simplex L which is correlative to the face in ν dimensions of parallelohedra characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i[,]n-\nu}).$$

One concludes that the number of incongruent faces in $n-\nu$ dimensions of the set (L) of simplexes is expressed by the symbol $\sigma_{n-\nu}$ $(\nu = 0, 1, 2, \ldots, n - 1)$.

As all the vertices of simplexes of the set (L) are congruent, one will declare $\sigma_0 = 1$, and the formula (9) of Number 59

$$S_\nu = (n + 1 - \nu)\sigma_{n-\nu}$$

will hold for the values of $\nu = 0, 1, 2, \ldots, n$, provided that one would admit $S_n = 1$.

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Theorem I. *The set (L) of simplexes uniformly fills the space in n dimensions.*

Let us suppose that a point (x_i) be interior to a face of the simplex L characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i\nu}). \tag{2}$$

One will have

$$x_i = \sum_{k=0}^{\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{\nu} \vartheta_k = 1 \text{ and } \vartheta_k > 0. \tag{3}$$

$$(k = 0, 1, 2, \ldots, \nu)$$

Let us suppose that the point (x_i) be interior to a face in ν' dimensions of another simplex L' characterised by the vertices

$$(l'_{i0}), (l'_{i1}), \ldots, (l'_{i\nu}).$$

One can write

$$x_i = \sum_{k=0}^{\nu'} \vartheta'_k l'_{ik} \text{ where } \sum_{k=0}^{\nu'} \vartheta'_k = 1 \text{ and } \vartheta'_k > 0.\, a; \qquad (4)$$

$$(k = 0, 1, 2, \ldots, \nu')$$

Let $\sum\sum a_{ij} x_i x_j$ be a positive quadratic form which defines a set (R) of primitive parallelohedra belonging to the type examined.

Let us indicate by (α_i) and (α'_i) two vertices of parallelohedra of the set (R) which are correlative to the simplexes L and L'. One will have the equalities

$$\begin{cases} \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A, \\ \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha'_i l'_{ik} = A'. \end{cases} \quad (k = 0, 1, 2, \ldots, n) \quad (5)$$

By putting down

$$\begin{cases} \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} = A + \rho_k, \\ \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha'_i l_{ik} = A' + \rho'_k, \end{cases} \quad (k = 0, 1, 2, \ldots, n) \quad (6)$$

one will have the inequalities

$$\rho_k \geq 0 \text{ and } \rho'_k \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

From equalities (5) and (6) one derives

$$\begin{cases} A' - A + 2\sum(\alpha_i - \alpha'_i) l'_{ik} = \rho_k, \\ A - A' + 2\sum(\alpha'_i - \alpha_i) l_{ik} = \rho'_k. \end{cases} \quad (k = 0, 1, 2, \ldots, n)$$

By virtue of equalities (3) and (4), one obtains,

$$A' - A + 2\sum(\alpha_i - \alpha'_i) x_i = \sum_{k=0}^{\nu'} \rho_k \vartheta'_k,$$

$$A - A' + 2\sum(\alpha'_i - \alpha_i) x_i = \sum_{i=0}^{\nu} \rho'_k \vartheta_k.$$

By making the sum of these equalities, one finds

$$\sum_{k=0}^{\nu'} \rho_k \vartheta'_k + \sum_{k=0}^{\nu} \rho'_k \vartheta_k = 0.$$

It follows, because of (3) and (4), that

$$\rho_k = 0, \ \ (k = 0, 1, 2, \ldots, \nu') \quad \text{and} \ \ \rho'_k = 0.$$
$$(k = 0, 1, 2, \ldots, nu)$$

let us notice that the equality $\rho_k = 0$ is possible only on the condition that the system (l'_{ik}) is found among the vertices of the simplex L, similarly, the equality $\rho'_k = 0$ is possible only on condition that the system (l_{ik}) is found among the vertices of the simplex L'.

One concludes that the systems

$$(l'_{i0}), (l'_{i1}), \ldots, (l'_{i\nu'}) \tag{7}$$

characterise a face of the simplex L and that the systems (2) characterise a face of the simplex L'. As a point (x_i) can not be interior to two different faces of the same simplex, it results in that the systems (2) and (7) coincide; therefore the two simplexes L and L' are contiguous through the faces in ν dimensions characterised by the systems (2).

It remains to demonstrate that any point (x_i) of the space in n dimensions belongs to at least one simplex of the set (L).

To demonstrate this, let us take any one point (ξ_i) which is interior to the simplex L and draw any one curve C which joins the points (ξ_i) and (x_i). I say that all the points of the curve C will be situated in the simplexes

$$L, L', \ldots, L^{(m)}$$

belonging to the set (L). In effect, let us suppose that the point (x_i) not belong to the simplex L. The curve C will go beyond in one point (ξ'_i) the boundary of the simplex L and will pass through a simplex L' which is contiguous to the simplex L through a face in any one number of dimensions and so on and so forth.

Theorem II. *A point* (x_i) *the elements* $x_1, x_2, \ldots, x_n$ *of which are integers can be only one vertex of simplexes of the set* (L).

Let us notice that there exist simplexes of the set (L) which passes the vertex (0); the number of these simplexes is expressed by the symbol S_0.

By effecting the translations of these simplexes the length of the vector $[x_i]$, one will obtain S_0 simplexes which possess the vertex (x_i). By virtue of Theorem I, the point (x_i) can not belong to other simplexes of the set (L).

Corollary. *Suppose that a point* (x_i) *the elements* $x_1, x_2, \ldots, x_n$ *of which are integers, is not found among the vertices*

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

of a simplex L. *By writing*

$$x_i = \sum_{i=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1,$$

one will have among the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ *at least one negative number.*

Properties of symbols S_ν *and* σ_ν $(\nu = 0, 1, 2, \ldots, n)$.

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Let us take any one positive integer m and consider a set K of points which are congruent to m^n points

$$\frac{g_1}{m}, \frac{g_2}{m}, \ldots, \frac{g_n}{m}$$

which one obtains by attributing to the numbers $g_1, g_2, \ldots, g_n$ the integer values verifying the inequalities

$$0 \leq g_k < m. \quad (k = 1, 2, \ldots, n)$$

Let us take any one point $(\frac{x_i}{m})$ of the set K and suppose that the point $(\frac{x_i}{m})$ be interior to any one face $P(\nu)$

of simplexes of the set (L) $(\nu = 0, 1, 2, \dots, n)$. By virtue of Theorem I of number 61, all points of the set K which are interior to the face $P(\nu)$ can not be congruent.

Let us indicate by

$$P_k^{(\nu)}, \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

the various incongruent faces of simplexes of the set (L) and by the symbol

$$m_k^{(\nu)} \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

let us indicate the number of points of the set K which are interior to the face $P_k(\nu)$. ONe will have a formula

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} m_k^{(\nu)} = m^n. \tag{1}$$

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It is easy to determine the value of the symbol $m_k^{(\nu)}$. Let us indicate by

$$l_{i0}^{(k)}, l_{i1}^{(k)}, \dots, l_{i\nu}^{(k)}$$

the vertices of the face $P_k(\nu)$ and let us write

$$\frac{x_i}{m} = \sum_{r=0}^{\nu} \vartheta_r l_{ir}^{(k)} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0.$$

$$(r = 0, 1, 2, \dots, \nu)$$

These equalities can be written

$$\frac{x_i}{m} - l_{i0}^{(k)} = \sum_{r=1}^{\nu} \vartheta_r (l_{ir}^{(k)} - l_{i0}^{(k)}).$$

By indicating, to make short,

$$x_i - m l_{i0}^{(k)} = t_i, \;\; l_{ir}^{(k)} - l_{i0}^{(k)} = p_{ir}, \quad (r = 1, 2, \dots, \nu)$$

and

$$m\vartheta_r = \tau_r, \quad (r = 1, 2, \ldots, \nu)$$

one will have

$$t_i = \sum_{r=1}^{\nu} \tau_r p_{ir} \text{ where } \sum_{r=1}^{\nu} \tau_r < m \text{ and } \tau_r > 0. \; (r = 1, 2, \ldots, \nu) \tag{2}$$

Let us indicate by $\omega_k^{(\nu)}$ the greatest common divisor of determinants of the order ν^2 which one can form from ν systems

$$(p_{i1}), (p_{i2}), \ldots, (p_{i\nu})$$

and suppose that the forms (2) represent the integers $t_1, t_2, \ldots, t_n$, provided that the numbers $\tau_1, \tau_2, \ldots, \tau_\nu$ belong to one of $\omega_k^{(\nu)}$ sets

$$\tau_r = \xi_{rh} + y_r \text{ where } r = 1, 2, \ldots, \nu, h = 1, 2, \ldots, \omega_k^{(\nu)}, \tag{3}$$

$y_1, y_2, \ldots, y_\nu$ being arbitrary integers.

One can suppose that

$$0 < \xi_{rh} \leq 1. \quad (r = 1, 2, \ldots, \nu, h = 1, 2, \ldots, \omega_k^{(\nu)})$$

By substituting the expressions of τ_r $(r = 1, 2, \ldots, \nu)$ derived from equalities (3) in the inequalities (2), one obtains

$$\xi_{rh} + y_r > 0, (r = 1, 2, \ldots, \nu) \quad \sum_{r=1}^{\nu} (\xi_{rh} + y_r) < m. \tag{4}$$

Let us indicate

$$\sum_{r=1}^{\nu} \xi_{rh} = a_h + \xi_h \tag{5}$$

where the integer a_h is determined after the conditions

$$0 \leq \xi_p < 1. \tag{6}$$

The inequalities (4) can be replaced by the following ones:

$$y_r \geq 0, \ (r = 1, 2, \ldots, \nu) \text{ and } \sum_{r=1}^{\nu} y_r \leq m - a_h - 1.$$

The number of systems $(y_1, y_2, \ldots, y_\nu)$ of integers $y_1, y_2, \ldots, y_\nu$ verifying these inequalities is equal to

$$\frac{(m - a_h)(m + 1 - a_h) \cdots (m + \nu - 1 - a_h)}{1 \cdot 2 \cdots \nu.}$$

By replacing with $a_{hk}^{(\nu)}$ the number a_h corresponding to the various sets (3), one obtains the formula

$$m_k^{(\nu)} = \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)})(m + 1 - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu}. \quad (7)$$

By substituting in the equality (1), one finds

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = m^n. \quad (8)$$

The formula obtained holds, whatever may be the positive integer value of m. One concludes this that this formula presents an identity.

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By comparing the coefficients of m^n in the formula (8), one finds

$$\sum_{k=1}^{\sigma_n} \omega_k^n = n!.$$

It follows that

$$\sigma_n \leq n!.$$

Let us introduce in our studies the finite difference of different orders by defining them by the formula

$$\Delta^{(\mu)} f(m) = \sum_{k=0}^{\mu} (-1)^{\mu-k} \frac{\mu!}{k!(\mu-k)!} f(m+\mu).$$

The formula (8) gives

$$\sum_{\nu=\mu}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m+\mu-a_{hk}^{(\nu)})\cdots(m+\nu-1-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)} = \Delta^{(\mu)}(m^n).$$

$$(\mu = 0, 1, 2, \ldots, n)$$

By making $m = 1$ in this formula and by noticing that

$$\frac{(\mu+1-a_{hk}^{(\nu)})\cdots(\nu-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)} \geq 0$$

since, because of (5) and (6)

$$a_{hk}^{(\nu)} \leq \nu,$$

one finds

$$\sum_{k=1}^{\sigma_\mu} \omega_k^{(\mu)} \leq \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \ldots, n) \qquad (9)$$

It follows that

$$\sigma_\mu \leq \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \ldots, n)$$

We have seen in Number 60 that

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}, \quad (\nu = 0, 1, 2, \ldots, n) \qquad (10)$$

therefore

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

Let us examine the conditions which have to be fulfilled for the symbols S_ν $(\nu = 0, 1, 2, \ldots, n)$ to be expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \ (\nu = 0, 1, 2, \ldots, n) \quad (11)$$

By virtue of inequalities (9), it is necessary that

$$\omega_k^{(\mu)} = 1. \ (k = 1, 2, \ldots, \sigma_\mu; \mu = 0, 1, 2, \ldots, n) \quad (12)$$

These are the conditions necessary and sufficient for the formula (11) to hold. In effect, in the case $\omega_k^{(\nu)} = 1$, the formula (7) becomes

$$m_k^{(\nu)} = \frac{(m-\nu)(m+1-\nu)\cdots(m-1)}{1\cdot 2\cdots \nu},$$

and the equality (8) takes the form

$$\sum_{\nu=0}^{n} \sigma_\nu \frac{(m-\nu)\cdots(m-1)}{1\cdot 2\cdots \nu} = m^n.$$

It follows that

$$\sum_{\nu=\mu}^{n} \sigma_\nu \frac{(m+\mu-\nu)\cdots(m-1)}{1\cdot 2\cdots (\nu-\mu)} = \Delta^{(\mu)}(m^n),$$

and by making $m = 1$, one obtains

$$\sigma_\mu = \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \ldots, n)$$

It results in, because of (10), the formula (11).

Let us notice that the conditions (12) come down to a single condition

$$\sigma_n = n!.$$

We will see that there exists primitive parallelohedra which satisfy this condition.

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Theorem. *The faces in 1, 2, 3 and 4 dimensions of simplexes of the set (L) enjoy the property that*

$$\omega^{(}_{k}\nu) = 1. \quad (k = 1, 2, \ldots, \sigma_\nu; \nu = 1, 2, 3, 4)$$

The demonstration of the theorem introduced does not present difficulties.

Corollary. *The number of faces in different dimensions of primitive parallelohedra in the space of 2, 3 and 4 dimensions is expressed by the formula (11).*

1. By making in the formula (11) $n = 2$, one obtains

$$S_0 = 6 \text{ and } S_1 = 6.$$

2. By making in the formula (11) $n = 3$, one obtains

$$S_0 = 24, \; S_1 = 36, \; S_2 = 14.$$

3. By making in the formula (11) $n = 4$, one obtains

$$S_0 = 120, \; S_1 = 240, \; S_2 = 150, \; S_3 = 30.$$

By studying the primitive parallelohedra in the space of 5 dimensions, I have come across parallelohedra the number of faces of which is not expressed by the formula (11).

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We have seen that, in the case $\omega_k^{(\nu)} = 1$, one has

$$a_{kh}^{(\nu)} = \nu.$$

It is easy to demonstrate that, in the case $\omega_k^{(\nu)} > 1$, one will have this equality for a single set (3) which is composed of integers; for all the sets which remain, one will have the inequalities

$$2 \leq a_{hk}^{(\nu)} \leq \nu - 2. \ (\nu \geq 5) \tag{13}$$

Let us make in the formula (8) $m = 0$. By noticing that

$$\frac{(-a_{hk}^{(\nu)})(1 - a_{hk}^{(\nu)}) \cdots (\nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = 0$$

so long as $a_{hk}^{(\nu)} \neq \nu$, one finds

$$\sum_{\nu=0}^{n} (-1)^{\nu} \sigma_{\nu} = 0.$$

By making in the formula (8) $m = -1$, one obtains, because of (13),

$$\sum_{\nu=0}^{n} (-1)^{\nu} (\nu + 1) \sigma_{\nu} = (-1)^{n}.$$

By substituting in this formula the expression of σ_{ν} derived from the formula (10), one will have

$$\sum_{\nu=0}^{n} (-1)^{\nu} S_{\nu} = 1. \tag{14}$$

Let us notice that the equality obtained expresses a property of faces in different dimensions of primitive parallelohedra which is common to all the convex polyhedra of the space in n dimensions. † By making in the formula (14) $n = 3$, one will have

$$S_0 - S_1 + S_2 - S_3 = 1,$$

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† See: *Poincaré*, *Sur la généralisation d'un théorème d'Euler relatif aux*

polyèdres. [On the generalisation of the theorem of Euler relative to the polyhedra] (*Comptes Rendus des Séances de l'Académie de Paris*, V. 117, p. 144) and as $S_3 = 1$, this becomes

$$S_0 + S_2 = 2 + S_1.$$

This is the well known formula of Euler. ‡

Regulators and characteristics of edges of primitive parallelohedra.

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Let us examine the set (R) of primitive parallelohedra belonging to a type of parallelohedra characterised by a set (L) of simplexes.

Let (α_i) be a vertex of parallelohedra of the set (R) determined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A. \ (k = 1, 2, \ldots, n) \qquad (1)$$

The system L correlative to the vertex (α_i) is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}). \qquad (2)$$

Let us indicate by $(\alpha_{ik}\ (k = 0, 1, 2, \ldots, n))$ the vertices adjacent to the vertex (α_i) (Number 18). The simplex L_k $(k = 0, 1, 2, \ldots, n)$ correlative to the vertex (α_{ik}) will be characterised by the systems which one obtains from system (2) by replacing the vertex (l_{ik}) of the simplex L by a corresponding vertex (l'_{ik}) of the simplex L_k. The two simplexes L and L_k are contiguous by a face in $n-1$ dimensions P_k $(k = 0, 1, 2, \ldots, n)$ which is characterised by the systems

$$(l_{ih}). \quad (k = 0, 1, 2, \ldots, n; h \neq k)$$

The face P_k of simplexes L and L_k is correlative to an edge $[\alpha_i, \alpha_{ik}]$ of parallelohedra of the set (R).

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‡ Euler, *Elementa doctrinae Solidorum.* (*Novi Comment. Petrop.* 1758.)

Let us indicate by

$$\sum p_{ik}x_i = \delta_k, \quad (k = 0, 1, 2, \ldots, n)$$

the equation of the face P_k. As one has

$$\sum p_{ik}l_{ik} = \delta_k, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

it becomes

$$\sum p_{ik}(l_{ik} - l_{ir}) = 0. \tag{3}$$

$$(h = 0, 1, 2, \ldots, n; r = 0, 1, 2, \ldots, n; h \neq k; r \neq k)$$

The equalities obtained define the number $p_{1k}, p_{2k}, \ldots, p_{nk}$ to a common factor close by. By supposing that $p_{1k}, p_{2k}, \ldots, p_{nk}$ be integer not having common divisor, one will determine by the equality (3) two systems (p_{ik}) and $(-p_{ik})$. One will call characteristic of the edge $[\alpha, \alpha_{ik}]$ or of the correlative face P_k one of the two systems $\pm(p_{ik})$ likewise.

By noticing that

$$\sum p_{ik}l_{ik} \neq \delta_k,$$

one will attach, for more precision, a supplementary condition

$$\sum p_{ik}l_{ik} > \delta_k.$$

Definition. *One will call characteristic of the face P_k with regard to the simplex L the system (p_{ik}) which is well defined by the conditions*

$$\sum p_{ik}l_{ik} > \delta_k, \sum p_{ik}l_{ih} = \delta_k. \ (h = 0, 1, 2, \ldots, n, h \neq k) \tag{4}$$

Let us notice that the characteristic of the face P_k with regard to the simplex L_k will be the system $(-p_{ik})$. In effect, one will have

$$\sum p_{ik}l'_{ik} \neq \delta_k.$$

Let us suppose that

$$\sum p_{ik}l'_{ik} > \delta_k.$$

In this case the two simplexes L and L_k would be situated on the same side of the face P_k, and one could find a point interior to the simplex L which would be interior to the simplex L_k too, this is contrary to Theorem I demonstrated in Number (6). It is therefore necessary that

$$\sum p_{ik} l'_{ik} < \delta_k,$$

and the system $(-p_{ik})$ presents the characteristic of the face P_k with regard to the simplex L_k.

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One will determine the vertex (α_{ik}) $(k = 0, 1, 2, \ldots, n)$ correlative to the simplex L_k by the equations

$$\sum\sum a_{ij} l_{ih} l_{jh} + 2\sum \alpha_{ik} l_{ih} = A_k \;\; (h = 0, 1, 2, \ldots, n; h \neq k) \tag{5}$$

and

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_{ik} l'_{ik} = A_k. \tag{6}$$

From the equalities (1) and (5), one derive

$$2\sum(\alpha_{ik} - \alpha_i) l_{ih} = A_k - A. \;\; (h = 0, 1, 2, \ldots, n; h \neq k) \tag{7}$$

As a result because of (3), one will have

$$\alpha_{ik} - \alpha_i = p_{ik} \rho_k. \;\; (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n) \tag{8}$$

On the ground of the supposition made, the vertices (α_i) and (α_{ik}) $(k = 0, 1, 2, \ldots, n)$ of primitive parallelohedra of the set (R) are simple.

It follows that,

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_{ik} l'_{ik} > A$$

and

$$\sum\sum a_{ij} l_{ik} l_{i[\text{sic}]k} + 2\sum \alpha_{ik} l_{ik} > A_k.$$

By virtue of (1) and (6), one obtains

$$2\sum(\alpha_{ik} - \alpha_i) l_{ik} > A_k - A \text{ and } 2\sum(\alpha_{ik} - \alpha_i) l'_{ik} < A_k - A$$

and, because of (8), it becomes

$$2\rho_k \sum p_{ik} l_{ik} > A_k - A \text{ and } 2\rho_k \sum p_{ik} l'_{ik} < A_k - A. \qquad (9)$$

As by virtue of (7) and (8), one has

$$2\rho_k \sum p_{ik} l_{ih} = A_k - A, \ \ (h = 0, 1, 2, \ldots, n; h \neq k) \qquad (10)$$

the inequalities (9) can be written

$$2\rho_k \sum p_{ik}(l_{ik} - l_{ih}) > 0, \ 2\rho_k \sum p_{ik}(l'_{ik} - l_{ih}) < 0. \qquad (11)$$
$$(h = 0, 1, 2, \ldots, n; h \neq k)$$

By noticing that because of (4)

$$\sum p_{ik}(l_i - l_{ih}) > 0, \ \ (h = 0, 1, 2, \ldots, n; h \neq k) \qquad (12)$$

one finds

$$\rho_k > 0, \ \ (k = 0, 1, 2, \ldots, n) \qquad (13)$$

and the second inequality (11) gives

$$\sum p_{ik}(l'_{ik} - l_{ih}) < 0, \ \ (h = 0, 1, 2, \ldots, n; h \neq k) \qquad (14)$$

or differently, because of (4),

$$\sum p_{ik} l'_{ik} < \delta_k, \ \ (k = 0, 1, 2, \ldots, n) \qquad (15)$$

that which we have demonstrated by another method.

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By substituting in (6) the expression of α_{ik} derived from the equality (8), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} + 2\rho_k \sum p_{ik} l'_{ik} = A_k.$$

One will present this equality, because of (1), under the form

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{0k} - \sum\sum a_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} =$$
$$A_k - A - 2\rho_k \sum p_{ik} l'_{ik},$$

and lastly, by virtue of (10),

$$\begin{gathered} 2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \\ \sum\sum a_{ij} l'_{ik} l_{jk} + 2\sum \alpha_i l'_{ik} - \sum\sum \alpha_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} \end{gathered} \qquad (16)$$

where $h = 0, 1, 2, \ldots, n, h \neq k, k = 0, 1, 2, \ldots, n.$

Definition II. *One will call regulator of the edge* $[\alpha_i, \alpha_{ik}]$ *or of the correlative face* P_k *the positive parameter* ρ_k *defined by the formulae (8) and (16).*

Let us notice that on the ground of equalities (3) and (8) the congruent edges and the congruent correlative faces possess the same regulator and the same characteristic.

One can determine the regulator ρ_k by other formulae.

Let us write

$$l'_{ik} = \sum_{r=0}^{n} \vartheta_r^{(k)} l_{ir} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \ (k = 1, 2, \ldots, n) \quad (17)$$

On the grounds of equations (1) and (17), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} - \sum\sum a_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr}.$$

By substituting in the formula (16), one finds

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr} \quad (18)$$
$$\text{where } h = 0, 1, 2, \ldots, n; h \neq k; k = 0, 1, 2, \ldots, n.$$

The formula obtained makes visible an important property of the regulator ρ_k: *the regulator* ρ_k *is expressed by a linear function of coefficients of the quadratic form* $\sum\sum a_{ij} x_i x_j$. By writing

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij}, \quad (19)$$

one will have the rational coefficients $p_{ij}^{(k)} = p_{ji}^{(k)}$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

By virtue of the formula (19), the regulator ρ_k will be perfectly determined if one knows the corresponding coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$.

As the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ of the quadratic form $\sum\sum a_{ij} x_i x_j$ do not play any role in the determination of coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ which depend only on the simplexes L and L_k, one can replace in the previous formula the coefficients a_{ij} by the coefficients $x_i x_j$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$.

By introducing the linear functions, as we have done in Number 58,

$$u_r = \sum l_{ir} x_i, \; v_r = \sum l'_{ir} x_i, \quad (r = 0, 1, 2, \ldots, n)$$

let us indicate by

$$u_r^{(k)} \text{ and } v_r^{(k)} \quad (r = 0, 1, 2, \ldots, n)$$

the values of these functions which correspond to the values of variables $x_1, x_2, \ldots, x_n$

$$x_i = p_{ik}. \quad (k = 0, 1, 2, \ldots, n)$$

By virtue of (4), one will have

$$u_h^{(k)} = \delta_k. \quad (h = 0, 1, 2, \ldots, n; h \neq k, u_k^{(k)} > \delta_k)$$

By virtue of (15), one will have

$$v_k^{(k)} < \delta_k.$$

Let us notice that the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ defined by the equality (7) will be determined by the equalities

$$v_k = \sum_{r=0}^{n} \vartheta_r^{(k)} u_r \text{ where } \sum \vartheta_r^{(k)} = 1.$$

By replacing in the formula (18) the coefficients a_{ij} by the coefficients $x_i x_j$, $i = 1, 2, \dots, n, j = 1, 2, \dots, n$, one obtains

$$2\rho_k(\delta_k - v_k^{(k)}) = (v_k)^2 - \sum_{r=0}^{n} \vartheta_r^{(k)}(u_r)^2. \ (k = 0, 1, 2, \dots, n) \quad 20$$

To return the formula obtained to the formula (19), it suffices to replace in the equality

$$\rho_k = \sum\sum p_{ij}^{(k)} x_i x_j$$

the coefficients $x_i x_j$ by the coefficients a_{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$.

Fundamental transformation of the form

$$\sum\sum a_{ij}x_i x_j + 2\sum \alpha_i x_i - \sum\sum a_{ij} l_i l_j - 2\sum \alpha_i l_i$$

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By keeping the previous notations, let us indicate

$$F_{(L)}(x_1, x_2, \dots, x_n) = \sum\sum a_{ij}x_i x_j + 2\sum \alpha_i x_i - A \qquad (1)$$

where one has admitted

$$A = \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik}. \ (k = 0, 1, 2, \dots, n) \qquad (2)$$

By introducing the variables $\xi_0, \xi_1, \dots, \xi_n$ after the conditions

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \qquad (3)$$

one will present the function $F_{(L)}$ $(x_1, x_2, \dots, x_n)$ under the following form:

$$F_{(L)}(x_1, x_2, \dots, x_n) = \sum\sum a_{ij}x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr}. \quad (4)$$

One concludes that the function $F_{(L)}\ (x_1, x_2, \ldots, x_n)$ is linear with regard to the coefficients $a_{ij}, i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ of an arbitrary quadratic form $\sum\sum a_{ij}x_ix_j$.

By making in the formula (1) $x_i = l_{ik}$, one obtains, because of (2),

$$F_{(L)}(l_{1k}, l_{2k}, \ldots, l_{nk}) = 0. \quad (k = 0, 1, 2, \ldots, n)$$

The equalities obtained hold, whatever may be the values of a_{ij} , $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Let us indicate by L^0 a simplex congruent to the simplex L and characterised by the vertices

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

By noticing that because of (3)

$$x_i + l_i = \sum_{r=0}^{n} \xi_r(l_{ir} + l_i) \text{ where } \sum_{r=0}^{n} \xi_r = 1,$$

one will have an equality

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij}(x_i + l_i)(x_j + l_j) \\ - \textstyle\sum_{r=0}^{n} \xi_r \sum\sum a_{ij}(l_{ir} + l_i)(l_{jr} + l_j)$$

and after the reductions, it becomes

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \\ \textstyle\sum\sum a_{ij}x_ix_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij}l_{ir}l_{jr},$$

therefore, because of (4) one will have

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = F_{(L)}(x_1, x_2, \ldots, x_n). \quad (5)$$

By virtue of the formula (18) of Number 71, one will determine the regulator ρ_k in the formula

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \quad (7)$$

$$(k = 0, 1, 2, \ldots, n)$$

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Let us indicate

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_{ik}x_i - A_k,$$

$$(k = 0, 1, 2, \ldots, n)$$

L_k being a simplex contiguous to the simplex L by the face P_k $(k = 0, 1, 2, \ldots, n)$.

By substituting in this equality the expression of α_{ik} defined from the formula (8) of Number 70, one obtains

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i + 2\rho_k \sum p_{ik}x_i - A_k$$

and, because of (1), one will have

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}x_i + A - A_k.$$

By substituting in this equality the expression of $A - A_k$ given by the formula (10) of Number 70, one finds

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_r \sum p_{ik}(x_i - l_{ih}).$$

$$(h \neq k)$$

This formula can be written

$$F_{(L)}(x_1, x_2, \ldots, x_n) = F_{L_k}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}(l_{ih} - x_i). \qquad (^*)$$

$$(h \neq k, k = 0, 1, 2, \ldots, n)$$

The formula $(^*)$ obtained is capable of numerous and important applications.

75

Let us suppose that $x_1, x_2, \ldots, x_n$ be arbitrary integers and that the point (x_i) is not found among the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \qquad (7)$$

of the simplex L. By admitting

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \tag{8}$$

one will have by virtue of Theorem II of Number 62 among the numbers $\xi_0, \xi_1, \ldots, \xi_n$ at least one negative number. Let us suppose to fix the ideas that

$$\xi_k < 0. \tag{9}$$

By noticing that because of (3) and of the formula (4) in Number 69 one has

$$\sum p_{ik}(l_{ih} - x_i) = \xi_k \sum p_{ik}(l_{ih} - l_{ik})$$

and that

$$\sum p_{ik}(l_{ih} - l_{ik}) < 0, \quad (h \neq k)$$

one obtains, because of (9),

$$\sum p_{ik}(l_{ih} - x_i) > 0.$$

One concludes that the coefficient of $2\rho_k$ in the formula (*) is an integer and positive in the case considered.

In the same manner, one will examine the function $F_{L_k}(x_1, x_2, \ldots, x_n)$ and so on.

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Let us suppose that one have examined the simplex,

$$L, L', L'', \ldots, L^{(m)} \tag{10}$$

successively contiguous by the faces in $n-1$ dimensions the regulator of which present the function

$$\rho_1, \rho_2, \ldots, \rho_m.$$

Let us suppose that by applying the formula (*) to the simplexes (10) one have obtained the equalities

$$\begin{aligned}
F_L(x_1, x_2, \ldots, x_n) &= F_{L'}(x_1, x_2, \ldots, x_n) + 2h_1\rho_1 \\
&\qquad \text{where } h_1 > 0, \\
F_{L'}(x_1, x_2, \ldots, x_n) &= F_{L''}(x_1, x_2, \ldots, x_n) + 2h_2\rho_2 \\
&\qquad \text{where } h_2 > 0, \\
&\ldots \\
F_{L^{(m-1)}}(x_1, x_2, \ldots, x_n) &= F_{L^{(m)}}(x_1, x_2, \ldots, x_n) + 2h_m\rho_m \\
&\qquad \text{where } h_m > 0.
\end{aligned}$$

It follows that

$$F_L(x_1, x_2, \ldots, x_n) = 2\sum_{k=1}^{m} h_k\rho_k + F_{L^{(m)}}(x_1, x_2, \ldots, x_n). \quad (11)$$

The procedure shown can not be prolonged indefinitely and one will always arrive at a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

To demonstrate this, let us notice that the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the formulae obtained are arbitrary.

Let us suppose that one have chosen the positive quadratic form $\sum\sum a_{ij}x_ix_j$ which defines a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

We have seen in Number 70 that one will have the inequalities

$$\rho_k > 0. \quad (k = 1, 2, \ldots, m)$$

By virtue of the definition of the function $F_L(x_1, x_2, \ldots, x_n)$, one will have an inequality

$$F_L(x_1, x_2, \ldots, x_n) > 0,$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, abstraction made from vertices (7) of the simplex L. It results in

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) \geq 0$$

and the formula (11), in the case considered, gives

$$F_L(x_1, x_2, \ldots, x_n) \geq 2 \sum_{k=1}^{m} h_k \rho_k.$$

As the coefficients h_k $(k = 1, 2, \ldots, m)$ are of positive integers and the regulators ρ_k $(k = 1, 2, \ldots, m)$ belong to a series of regulators corresponding to the incongruent faces of simplexes of the set (L), one concludes that the number m can not be increased indefinitely. As a result the series (10) will be terminated by a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

It follows that one will have identically

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) = 0,$$

and the formula (11) becomes

$$F_L(x_1, x_2, \ldots, x_n) = 2 \sum_{[k]=1}^{m} h_k \rho_k \text{ where } (k = 1, 2, \ldots, m) \tag{12}$$

Let us notice that the formula obtained presents an identity which holds, whatever the values of coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ may be, provided that the regulators ρ_k $(k = 1, 2, \ldots, m)$ are expressed by the formula (6).

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Fundamental theorem. *Let us suppose that the regulators ρ_k $(k = 1, 2, \ldots, \sigma)$ corresponding to the various incongruent faces in $n-1$ dimensions of simplexes belonging to the set (L) be determined by the equations*

$$\rho_k = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}^{(k)} a_{ij}. \quad (k = 1, 2, \ldots, \sigma)$$

For a quadratic form $\sum\sum a_{ij}x_i x_j$ to define a set (R) primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the inequalities

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} > 0, \quad (k = 1, 2, \ldots, \sigma)$$

hold.

We have seen in Number 70 that the inequalities

$$\rho_k > 0, \ \ (k = 1, 2, \dots, \sigma) \qquad 13$$

present the necessary conditions. Let us suppose the coefficients of a quadratic form $\sum\sum a_{ij}x_ix_j$ verify the inequalities (13). By virtue of the formula (12), one will have the inequality

$$F_{(L)}(x_1, x_2, \dots, x_n) > 0$$

so long as the point (x_i) the element s of which are integers is not found among the vertices of the simplex L. By virtue of the definition established, the simplex L is in this case correlative to a simplex vertex (α_i) of parallelohedra corresponding to the quadratic form examined $\sum\sum a_{ij}x_ix_j$.

The simplex L is chosen arbitrary in the set (L) of simplexes, therefore all the simplexes of the set (L) are correlative to the simple vertices of parallelohedra corresponding to the quadratic form $\sum\sum a_{ij}x_ix_j$.

I argue that these parallelohedra do not possess other vertices, it is that which one will verify without trouble.

78

Let us notice that any quadratic form $\sum\sum a_{ij}x_ix_j$ verifying the inequalities (13) is positive. To demonstrate this, let us examine a simplex L among the vertices in which is found the point (0). One will have in this case

$$F_{(L)}(x_1, x_2, \dots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i,$$

and consequently

$$F_{(L)}(x_1, x_2, \dots, x_n) + F_{(L)}(-x_1, -x_2, \dots, -x_n) = 2\sum\sum a_{ij}x_ix_j.$$

The two points (x_i) and $(-x_i)$ can not be the vertices of the simplex L, the point (0) being excluded. This results in

$$F_{(L)}(x_1, x_2, \dots, x_n) + F_{(L)}(-x_1, -x_2, \dots, -x_n) > 0,$$

therefore

$$\sum\sum a_{ij}x_ix_j > 0,$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$, the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ being excluded.

Definition of quadratic forms with the help of regulators and corresponding characteristics.

79

Let us take any one quadratic form $\sum\sum a_{ij}x_ix_j$ in arbitrary coefficients. Let us choose n numbers $x_1, x_2, \ldots,$ x_n which are subject to the only condition: *the equality*

$$h_1x_2 + h_2x_2 + \ldots + h_nx_n = 0$$

is impossible so long as the numbers $h_1, h_2, \ldots, h_n$ are integers.

Let us examine a vector g made up of points

$$\frac{l_1}{m} + ux_i \text{ where } 0 \leq u \leq 1,$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers and m being any one positive integer.

The vector g will traverse a certain number of simplexes belonging to the set (L). Let us indicate by

$$L_0, L_1, \ldots, L_s \tag{1}$$

the simplexes of the set (L) which contain the various parts of the set g. On the ground of the supposition made, the simplexes (1) are well defined by the vector g and are successively contiguous by the faces in $n-1$ dimensions. In effect, two adjacent simplexes L_k and L_{k+1} of the series (1) possess a common point (ξ_{ik}) belonging to the vector g, therefore the simplexes L_k and L_{k+1} are contiguous by a face in any number of dimensions. Let us suppose that this face be characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i\nu}). \tag{2}$$

As

$$\xi_{ik} = \frac{l_i}{m} + u_k x_i \text{ where } 0 < u_k < 1,$$

one will have

$$\frac{l_i}{m} + u_k x_i = \sum_{r=0}^{\nu} \vartheta_r l_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \qquad (3)$$

$$(r = 0, 1, 2, \ldots, \nu)$$

By supposing that $\nu < n - 1$, one wil determine with the help of these equalities a system (h_i) of integers verifying the equation

$$h_1 x_1 + h_2 x_2 + \ldots + h_n x_n = 0,$$

which is contrary to the hypothesis, therefore it is necessary that $\nu = n - 1$ and that the point (ξ_{ik}) be interior to a face in $n - 1$ dimensions which is common to the simplexes L_k and L_{k+1}.

Let us suppose that $\nu = n - 1$. By indicating with (p_{ik}) the characteristic of the face (P_k) characterised by the systems (2) with regard to the simplex L_k, one will have, by virtue of the formula (4) of Number 69,

$$\sum p_{ik} l_{ir} = \delta_k, \quad (r = 0, 1, 2, \ldots, n - 1)$$

and the equalities (3) give

$$\sum p_i \left(\frac{l_i}{m} + u_k x_i\right) = \delta_k$$

and consequently

$$u_k \sum p_{ik} x_i = \delta - \sum p_{ik} \frac{l_i}{m}.$$

As $\sum p_{ik} x_i \neq 0$, on the ground of the supposition made, the equality obtained defines a point (ξ_{ik}) of the vector g which is interior to the face P_k. This results in that the vector g does not possess other points common to the face P_k. By attributing to the parameter u a negative variation δu sufficiently small, one will define a point $\frac{l_i}{m} + (u_k + \delta u)x_i$ of the vector g which is interior to the simplex L_k. By attributing to the parameter u a variable $\delta u > 0$, one obtains a point $\frac{l_i}{m} + (u_k + \delta u)x_i$ which is interior to the simplex L_{k+1}.

As in these cases one has

$$\sum p_{ik}(\frac{l_i}{m} + (u_k + \delta u)x_i) > \delta_k, \quad (\delta u < 0)$$

and

$$\sum p_{ik}(\frac{l_i}{m} + (u_k + \delta u)x_i) < \delta_k, \quad (\delta u > 0)$$

it becomes

$$\sum p_{ik}x_i < 0. \tag{4}$$

By indicating with ρ_k the regulator of the face P_k with regard to the chosen quadratic form $\sum a_{ij}x_ik_j$ $(k = 0, 1, 2, \dots, s-1)$, let us apply the formula (*) of Number 74 to the simplexes (1). One will have the equalities

$$\begin{aligned} F_{(L_0)}(\frac{l_1}{m} + x_1, \dots, \frac{l_n}{m} + x_n) &= F_{L_1}(\frac{l_1}{m} + x_1, \dots, \frac{l_n}{m} + x_n) \\ &+ 2\rho_0 \sum p_{i0}(l_{ih_0} - \frac{l_i}{m} - x_i), \\ &\dots \\ F_{L_{s-1}}(\frac{l_1}{m} + x_1, \dots, \frac{l_n}{m} + x_n) &= F_{L_s}(\frac{l_1}{m} + x_1, \dots, \frac{l_n}{m} + x_n) \\ &+ 2\rho_{s-1} \sum p_{i,s-1}(l_{ih_{s-1}} - \frac{l_i}{m} - x_i). \end{aligned}$$

It follows that

$$\begin{aligned} F_{(L_0)}(\tfrac{l_1}{m} + x_1, \dots, \tfrac{l_n}{m} + x_n) = 2\textstyle\sum_{k=0}^{s-1} \rho_k \sum_{i=1}^{n} p_{ik}(l_{ih_k} - \tfrac{l_i}{m} - x_i) \\ + F_{L_s}(\tfrac{l_1}{m} + x_1, \dots, \tfrac{l_n}{m} + x_n). \end{aligned} \tag{5}$$

Until now, the integers $l_1, l_2, \dots, l_n$ had been arbitrary. Let us suppose that the integers $l_1, l_2, \dots, l_n$ satisfy the conditions

$$0 \le l_i < m. \ (i = 1, 2, \dots, n) \tag{6}$$

Let us indicate by K the set of incongruent points $(\frac{l_i}{m})$ verifying these inequalities.

The number of points belonging to the set K is equal to m.

Let us apply the formula (5) to all the points of the set K and make the sum of equalities obtained. One will have a formula

$$F_{(L_0)}(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n) =$$
$$2\sum \rho_k \sum p_{ik}(l_{ih_k} - \frac{l_i}{m} - x_i) + \sum F_{(L_s)}(\frac{l_1}{m}+x_i,\dots,\frac{l_n}{m}+x_n). \tag{7}$$

All the sums which are formed in this formula can be determined with a certain approximation.

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Let us suppose that the simplex L_0 be characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{in}).$$

On the ground of the supposition made, the point $(\frac{l_i}{m})$ belongs to the simplex L_0. As there exist only a finite number of simplexes of the set L to which belong the points $(\frac{l_i}{m})$ verifying the inequalities (6), one concludes that one can determine a positive parameter λ in such a manner that the inequalities

$$|l_{ik}| \leq \lambda \ \ (i = 1, 2, \dots, n; k = 0, 1, 2, \dots, n) \tag{8}$$

holds.

In this case, the corresponding value of the function

$$F_{(L_0)}\left(\frac{l_1}{m}+x_1, \frac{l_2}{m}+x_2, \dots, \frac{l_n}{m}+x_n\right)$$

can be presented under the form

$$F_{(L_0)}\left(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n\right) = \sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i$$

where the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ do not exceed in numerical value a fixed limit ϵ which does not depend on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes.

Let us examine the function $F_{(L_s)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right)$. After the proposition made, the point $\left(\frac{l_i}{m} + x_i\right)$ belongs to the simplex L_s. Let us determine the integers $t_1, t_2, \ldots, t_n$ after the conditions

$$0 \leq \frac{l_i}{m} + x_i + t_i < 1, \;\; (i = 1, 2, \ldots, m) \tag{9}$$

and indicate by L'_s the simplex congruent to the simplex L_s which obtains by a translation of the simplex L_s the length of the vector $[t_i]$.

By virtue of the formula (5) of Number 73, one will have

$$\begin{aligned} F_{(L_s)}\left(\tfrac{l_1}{m} + x_1, \ldots, \tfrac{l_n}{m} + x_n\right) = \\ F_{L'_s}\left(\tfrac{l_1}{m} + x_1 + t_1, \ldots, \tfrac{l_n}{m} + x_n + t_n\right). \end{aligned} \tag{10}$$

By indicating with

$$(l'_{i0}), \ldots, (l'_{in})$$

the vertices of the simplex L'_s, one will have, by virtue of (9), the inequalities (8):

$$|l'_{ik}| \leq \lambda. \quad (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n)$$

It follows that the numerical value of the function

$$F_{L'_s}\left(\frac{l_1}{m} + x_1 + t_1, \ldots, \frac{l_n}{m} + x_n + t_n\right),$$

because of (8) and (9), does not exceed a fixed limit φ which depends only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes. By virtue of (10), one can write

$$F_{(L_s)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right) = \varphi_0 \text{ where } |\varphi_0| \leq \varphi.$$

By substituting in the formula (7) the results obtained, one will present it under the following form

$$\begin{aligned} m_n\left(\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_ix_i\right) = \\ 2\sum\rho_k\sum_{i=1}^{n}p_{ik}\left(l_{ih}-\tfrac{l_i}{m}-x_i\right) \end{aligned} \tag{11}$$

In this formula, the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ do not exceed in numerical value a fixed limit which does not depend on numbers $x_1, x_2, \ldots, x_n$.

81

Let us determine the coefficients of $2\rho_k$ in the formula obtained.

To that effect, let us choose any one face P in $n-1$ dimensions of simplexes of the set (L). Let us suppose that the face P is characterised by the systems

$$(0), (l_{(i1)}), \ldots, (l_{(i,n-1)}).$$

Let us indicate by ρ the regulator and by $\pm(p_i)$ the characteristic of the face P with regard to the quadratic form $\sum\sum a_{ij}x_ix_j$. One will suppose, on the ground of (4), that

$$\sum p_ix_i < 0. \tag{12}$$

Let us suppose that the vector g is made up of points

$$\frac{l_i}{m}+ux_i \text{ where } 0 \leq u \leq 1$$

and corresponding to a point $\left(\frac{l_i}{m}\right)$ belonging to the set K possesses a point which is interior to a face P' congruent to the face P.

By supposing that the face P' is characterised by the systems

$$(g_i), (l_{i1}+g_i), \ldots, (l_{i,n-1}, g_i),$$

one will have, on the ground of the supposition made,

$$\frac{l_i}{m}+ux_i = \sum_{k=0}^{n-1}\vartheta_k(g_i+l_{ik}) \text{ where } \sum_{k=0}^{n-1}\vartheta_k = 1 \text{ and } \vartheta_k > 0.$$

The corresponding value of the coefficient of 2ρ in the formula (11) is expressed by the sum

$$\sum \sum_{i=1}^{n} p_i (l_{ih} - \frac{l_i}{m} - x_i) \tag{14}$$

which extends to all the faces P' congruent to the face P verifying the equalities (13), provided that the points $(\frac{l_i}{m})$ belong to the set K.

Let us indicate, to make short,

$$\sum_{i=1}^{n} p_i x_i = -\Delta \tag{15}$$

One will have, because of (12),

$$\Delta > 0.$$

As the system (l_{ih}) in the sum (14) indicate any one vertex of the face P', one can write down

$$l_{ih} = g_i,$$

and the equalities (13) and (15) give

$$\sum_{i=1}^{n} p_i \left(g_i - \frac{l_i}{m} - x_i \right) = (1 - u)\Delta.$$

Therefore, the study of the coefficient of 2ρ in the formula (11) comes down to the evaluation of the sum

$$\sum (1 - u)\Delta \text{ where } 0 < u < 1. \tag{16}$$

82

Let us designate, to summarise,

$$-mg_i + l_i = h_i. \tag{17}$$

The parameter u verifying the equalities (13) is expressed by the formula

$$u = \frac{1}{m\Delta}\sum p_i h_i, \tag{18}$$

and as $0 < u < 1$, it becomes

$$0 < \sum p_i h_i < m\Delta.$$

By indicating with τ the integer verifying the inequalities

$$0 < \tau < m\Delta, \tag{19}$$

let us write

$$\sum p_i h_i = \tau. \tag{20}$$

By virtue of (18), the corresponding value of the parameter u will be

$$u = \frac{\tau}{m\Delta}.$$

Let us substitute the expression found of the parameter u in the equalities (13), it will become, because of (17),

$$\frac{\tau}{\Delta}x_i + h_i = m\sum_{k=1}^{n-1}\vartheta_k l_{ik} \text{ where } \sum_{k=1}^{n-1}\vartheta_k < 1 \text{ and } \vartheta_k > 0. \tag{21}$$

$$(k = 1, 2, \ldots, n-1)$$

This stated, let us notice that one can attribute to the number τ an arbitrary value verifying the inequalities (19). For similar values of τ to exist, it is necessary that

$$m\Delta > 1,$$

Let us suppose that the positive integer m satisfies this condition. In this case, the finding of the sum (16) comes down to the solution of a sum

$$\sum(1-u)\Delta = \sum_{\tau>0}^{\tau<m\Delta} m_\tau(\Delta - \frac{\tau}{m}) \tag{22}$$

where m_τ indicates the number of systems (h_i) of integers $h_1, h_2, \ldots, h_n$ verifying the equalities (20) and (21).

83

It is easy to determine the number m_τ.

Let us indicate by (h_i^0) a system of integers verifying the equality

$$\sum p_i h_i^0 = 1. \tag{23}$$

As, on the ground of the supposition made, the integers $p_1, p_2, \ldots, p_n$ have no common divisor, the systems of integers verifying this equality always exist.

One will determine all the systems (h_i) of integers verifying the equality (20) with the help of formulae

$$h_i = \tau h_i^0 + \sum_{k=1}^{n-1} \tau_k l_{ik} \tag{24}$$

where the rational numbers $\tau_1, \tau_2, \ldots, \tau_{n-1}$ belong to certain sets

$$\tau_k = \xi_{kr} + y_k, \quad (k = 1, 2, \ldots, n-1; r = 1, 2, \ldots, \omega) \tag{25}$$

ω being the greatest common divisor of n determinants of the order $(n-1)^2$ which one can form from $n-1$ systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i,n-1}).$$

By substituting the expressions of $h_1, h_2, \ldots, h_n$ derived from equalities (24) in the equalities (21), one obtains

$$\frac{\tau}{\Delta} x_i + \tau h_i^0 = \sum_{k-1}^{n-1} (m\vartheta_k - \tau_k) l_{ik}. \tag{26}$$

Let us notice that the numerical value of the determinant of n systems

$$(x_i), (l_{i1}), \ldots, (l_{i,n-1})$$

is expressed by the formula

$$\pm\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ l_{11} & l_{21} & \cdots & l_{n1} \\ & \cdots & \cdots & \\ l_{1,n-1} & l_{2,n-1} & \cdots & l_{n,n-1} \end{vmatrix} = -\sum x_i p_i \omega = \omega\Delta.$$

Let us indicate by $\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk}$, $(k = 1, 2, \ldots, n-1)$ the minor determinants which are defined by the equalities

$$\sum_{i=1}^{n} \lambda_{ik} l_{ik} = \omega\Delta, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum \lambda_{ik} x_i = 0, \sum \lambda_{ik} l_{ir} = 0. \quad (r = 1, 2, \ldots, n-1; r \neq k)$$

The equalities (26) give

$$\tau \sum \lambda_{ik} h_i^0 = \omega\Delta(m\vartheta_k - \tau_k), \quad (k = 1, 2, \ldots, n-1)$$

and as a result

$$m\vartheta_k = \tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0. \quad (k = 1, 2, \ldots, n-1)$$

By virtue of (21) one obtains the inequalities

$$\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 > 0, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum_{k=1}^{n-1} (\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0) < m.$$

Considering the set (25), one finds

$$\left\{ \begin{array}{l} y_k + \xi_{kr} + \dfrac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 > 0, \\ \displaystyle\sum_{k=1}^{n-1} \left(y_k \xi_{kr} + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 \right) < m. \end{array} \right. \quad (k = 1, 2, \ldots, n-1) \quad (27)$$

Let us write

$$y_k \xi_{kr} + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 = y_k' + \nu_k \text{ where } 0 < \nu_k \leq 1$$

$$(k = 1, 2, \ldots, n-1)$$

and

$$\sum_{k=1}^{n-1} \nu_k = a_r^{(\tau)} + \nu \text{ where } 0 \leq \nu < 1, \tag{28}$$

$y'_1, \ldots, y'_{n-1}$ and $a_r^{(\tau)}$ being integers.

The inequalities (27) will be replaced by the following ones:

$$\sum_{k=1}^{n-1} y'_k < m - a_r^{(\tau)} - \nu,\ y'_k > -\nu_k, \quad (k = 1, 2, \ldots, n-1)$$

or differently

$$\sum_{k=1}^{n-1} y'_k \leq m - a_r^{(\tau)} - 1 \text{ and } y'_k \geq 0. \quad (k = 1, 2, \ldots, n-1)$$

The number of systems $(y'_1, y'_2, \ldots, y'_{n-1})$ of integers verifying these inequalities is equal to

$$\frac{(m - a_r^{(\tau)})(m + 1 - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}.$$

One concludes that the symbol m_τ which expresses the number of solutions of equations (20) and (21) in integers is equal to the sum

$$m_\tau = \sum_{\tau=1}^{\omega} \frac{(m - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}.$$

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By substituting in the sum (26), one obtains

$$\begin{gathered} \sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \tfrac{\tau}{m}\right) m_\tau = \\ \sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \tfrac{\tau}{m}\right) \sum_{r=1}^{\omega} \tfrac{(m - a_r^{(\tau)}) \cdots (m+n-2-a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}. \end{gathered} \tag{29}$$

Let us find a value approached by the sum obtained. By noticing that because of (28)

$$0 \leq a_r^{(\tau)} \leq n - 1,$$

one can write

$$\sum_{r=1}^{\omega} \frac{(m - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)} = \frac{\omega}{(n-1)!} m^{n-1} + \delta_\tau m^{n-2}$$

where $|\delta_t au|$ does not exceed a fixed limit which does not depend on the number m.

By substituting in the sum (27), one finds

$$\sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \frac{\tau}{m}\right) m_\tau = \frac{\omega}{(n-1)!} \sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \frac{\tau}{m}\right) + \delta\Delta^2 m^{n-1}$$

where $|\delta|$ does not exceed a fixed limit which does not depend on $x_1, x_2, \ldots, x_n$ and on the number m.

By noticing that

$$\sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \frac{\tau}{m}\right) = \Delta^2 \frac{m}{2} - \frac{\Delta}{2} + \frac{\vartheta}{m} \text{ where } 0 \leq \vartheta < \frac{1}{8},$$

one can write

$$\sum_{\tau>0}^{\tau<m\Delta} \left(\Delta - \frac{\tau}{m}\right) m_\tau = \frac{1}{2} \frac{\omega\Delta^2}{(n-1)!} m^n + m^{n-1}(\delta\Delta^2 + \delta'\Delta + \delta'') \tag{30}$$

where $\delta, \delta', \delta''$ do not exceed in numerical value a fixed limit.

By substituting in the formula (11) the coefficient found of 2ρ, one will have, because of (19),

$$\begin{gathered} m^n\left(\sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i\right) = \\ \frac{m^n}{(m-1)!} \sum_{k=1}^{\sigma} \rho_k \omega_k \left(\sum p_i x_i\right)^2 + \\ m^{n-1} \sum_{k=1}^{\sigma} 2\rho_k \left[\delta_k \left(\sum p_{ik} x_i\right)^2 + \delta'_k \sum p_{ik} x_i + \delta''_k\right]. \end{gathered} \tag{31}$$

In the formula obtained the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$, $\delta_k, \delta'_k, \delta''_k$ $(k = 1, 2, \ldots, \sigma)$ do not exceed in numerical value a fixed limit which depend only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ snf on the choice of the set (L) of simplexes.

Let us replace in the formula obtained the numbers $x_1, x_2, \ldots, x_n$ by the numbers $mx_1, mx_2, \ldots, mx_n$. As these numbers satisfy the conditions imposed on the numbers $x_1, x_2, \ldots, x_n$, the formula (31) is applicable and one obtains

$$\begin{aligned} m^n(m^2 \textstyle\sum\sum a_{ij}x_ix_j + \epsilon_0 + m \sum \epsilon_i x_i) = \\ \tfrac{m^{n-2}}{(n-1)!} \textstyle\sum_{k=1}^{\sigma} \rho_k \omega_k (\sum p_{ik}x_i)^2 \\ + m^{n-1} \textstyle\sum_{k=1}^{\sigma} 2\rho_k [\delta_k m^2 (\sum p_{ik}x_i)^2 \\ + \delta'_k m \textstyle\sum p_{ik}x_i + \delta''_k]. \end{aligned}$$

By dividing the two parts of the formula obtained by m^{n+2}, let us make the positive integer m increase indefinitely, it will become

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!} \sum_{k=1}^{\sigma} \rho_k \omega_k (p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{nk}x_n)^2. \tag{32}$$

The sum which is found in the second member of the formula obtained extends to all the incongruent faces in $n-1$ in $n-1$ dimensions of simplexes of the set (L).

We have deduced the formula (34) by supposing that the numbers $x_1, x_2, \ldots, x_n$ form a irreducible basis. As the two parts of the formula (32) present two quadratic forms, one concludes that the formula (32) present an identity. This results in that the formula (32) can be written

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a'_{ij} = \frac{1}{(n-1)!} \sum_{k=1}^{\sigma} \rho_k \omega_k \sum_{i=1}^{n}\sum_{j=1}^{n} a'_{ij} p_{ik} p_{jk} \tag{I}$$

where one has written

$$\rho_k = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^{(k)} a_{ij}, \quad (k = 1, 2, \ldots, \sigma)$$

the two quadratic forms $\sum\sum a_{ij}x_ix_j$ and $\sum\sum a'_{ij}x_ix_j$ being arbitrary.

Section V.

Properties of the set (Δ)

of quadratic forms corresponding to
the various types of primitive parallelohedra.

Definition of the domain of quadratic form corresponding to a type of primitive parallelohedra.

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Let us suppose that a type of primitive parallelohedra is characterised by a set (L) of simplexes.

Let us indicate by

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \quad (k = 1, 2, \ldots, \sigma)$$

the regulators which correspond to the various incongruent faces in $n-1$ dimensions of simplexes of the set (L).

Definition. *One will call domain of quadratic forms corresponding to the type of primitive parallelohedra characterised by the set (L) of simplexes a domain Δ in quadratic forms verifying the inequalities*

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \geq 0. \ (k = 1, 2, \ldots, \sigma) \qquad (1)$$

On the ground of the fundamental theorem of Number 77, for a quadratic form f to define a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the form f is interior to the domain Δ. This results in that the domain Δ is of $\frac{n(n+1)}{2}$ dimensions.

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Among the inequalities (1) can be found dependent inequalities. Let us suppose that one has chosen a system of independent inequalities

$$\rho_1 \geq 0, \rho_2 \geq 0, \ldots, \rho_m \geq 0$$

which define the domain Δ. With the help of independent regulators $\rho_1, \rho_2, \ldots, \rho_m$ one will present all the regulators under the form

$$\rho_k = \sum_{r=1}^{m} h_r^{(k)} \rho_r \text{ where } h_r^{(k)} \geq 0. \qquad (2)$$

$$(r = 1, 2, \ldots, m; k = 1, 2, \ldots, \sigma)$$

Let us observe that any one quadratic form $\sum\sum a_{ij}x_ix_j$ does not verify the equations

$$\rho_1 = 0, \rho_2 = 0, \ldots, \rho_m = 0$$

because the equalities (2) give

$$\rho_k = 0, \quad (k = 1, 2, \ldots, \sigma)$$

and, by virtue of the formula (I) of Number 84, one has

$$\sum\sum a_{ij}a'_{ij} = 0,$$

$\sum\sum a'_{ij}x_ix_j$ being an arbitrary form; it follows that

$$a_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

To the domain Δ, therefore, the conclusion deduced in my first mémoire cited † are applicable.

Let us indicate by

$$\varphi_1, \varphi_2, \ldots, \varphi_s \tag{3}$$

the quadratic forms which characterise the various edges of the domain Δ.

The domain Δ of quadratic forms will be determined by the equalities

$$\sum\sum a_{ij}x_ix_j = \sum_{u=1}^{s} u_k\varphi_k \text{ where } u_k \geq 0, \quad (k = 1, 2, \ldots, s)$$

$u_1, u_2, \ldots, u_s$ being positive arbitrary parameters or zeros.

Let us notice that by virtue of the formula (I) of Number (84), each form $\varphi_k (k = 1, 2, \ldots, s)$ of the series (3) will have for expression

$$\varphi_k = \sum_{r=1}^{\sigma} \lambda_r^{(k)}(p_{1r}x_1 + p_{2r}x_2 + \ldots + p_{nr}x_n)^2$$

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† This journal V. 133, p. 97

where $\lambda_r^{(4)} \geq 0. \quad (r = 1, 2, \ldots, \sigma; k = 1, 2, \ldots, \sigma)$

Properties of independent regulators

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By keeping the notations from Number 69–74 let us suppose that a simplex L of the set (L) is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

Let us suppose that among the regulators

$$\rho_0, \rho_1, \ldots, \rho_n$$

which correspond to the various faces in $n-1$ dimensions of the simplex L, is found at least one independent regulator. Let us suppose, to fix the ideas, that ρ_0 is a similar regulator.

Let us indicate by L_k $(k = 0, 1, 2, \ldots, n)$ the simplexes which are contiguous to the simplex (L) through the faces in $n-1$ dimensions characterised by the systems

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = 0, 1, 2, \ldots, n)$$

Let us suppose that by replacing in the simplex L the vertex (l_{ik}) by a vertex (l'_{ik}), one obtains the simplex L_k $(k = 0, 1, 2, \ldots, n)$.

By virtue of the formula (6) of Number 73, one will have

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \qquad (1)$$

$$(k = 0, 1, 2, \ldots, n; h \neq k)$$

Let us admit

$$l'_{ik} = \sum_{k=0}^{n} \vartheta_r^{(k)} l_{ik} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \ (k = 0, 1, 2, \ldots, n) \quad (2)$$

As, because of the inequality (14) of Number 70,

$$\sum p_{ik}(l_{ih} - l'_{ik}) > 0, \quad (h \neq k)$$

it becomes, by virtue of (2)

$$\vartheta_k^{(k)} < 0. \quad (k = 0, 1, 2, \ldots, n) \tag{3}$$

Let us examine the numbers

$$\vartheta_0^0, \vartheta_1^0, \ldots, \vartheta_n^0 \tag{4}$$

which correspond to the independent regulator ρ_0. One will have, because of (3),

$$\vartheta_0^0 = 0.$$

I say that among the numbers $\vartheta_1^0, \vartheta_2^0, \ldots, \vartheta_n^0$ at least two numbers are positive. As $\sum_{k=0}^n \vartheta_k^0 = 1$, it is evident that at least one number, for example ϑ_n^0, will be positive. Let us suppose that ϑ_n^0 is the only positive number in the series (4).

Let us indicate, to fix the ideas,

$$\vartheta_0^0 < 0, \vartheta_1^0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda[+1]}^0 = 0, \ldots, \vartheta_{n-1}^0 = 0, \vartheta_n^0 > 0. \tag{5}$$

The corresponding value of the function $F_{(L)}(l'_{10}, \ldots, l'_{n0})$, by virtue of the formula (4) of Number 78, can be presented under the form

$$\begin{aligned} F_{(L)}(l'_{10}, \ldots, l'_{n0}) &= \sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) \\ &\quad - \sum_{k=0}^{\lambda} \vartheta_k \sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}). \end{aligned} \tag{6}$$

By virtue of the formula (I) of Number 84 and of inequalities (5), one can present this equality under the form

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = \sum_{r=1}^{\sigma} h_r \rho_r \text{ where } h_r \geq 0,$$

and as on the other hand, because of (1)

$$2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{i0}, \ldots, l'_{n0}), \tag{7}$$

it becomes

$$\rho_0 = \sum_{r=1}^{\sigma} g_r \rho_r \text{ where } g_r \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

We have supposed that ρ_0 is an independent regulator, therefore it is necessary that

$g_2 = 0$ so long as a regulator ρ_r is not proportional to ρ_0.

The formula (6) gives

$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \delta\rho_0 \text{ where } \delta > 0,$$

$$\sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}) = \delta_k\rho_0 \text{ where } \rho_k > 0.$$

$$(k = 0, 1, 2, \ldots, \lambda)$$

It follows that one has identically,

$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \frac{\delta}{\delta_k}\sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}).$$

For this identity to hold, it is necessary and sufficient that

$$l'_{i0} - l_{in} = \sqrt{\frac{\delta}{\delta_k}}(l_{ik} - l_{in}).$$

By virtue of Theorem I of Number 51, the numbers $l'_{i0} - l_{in}$ $(i = 1, 2, \ldots, n)$, and $l_{ik} - l_{in}$ $(i = 1, 2, \ldots, n)$, do not have common divisor, one concludes that

$$l'_{i0} = l_{ik},$$

which is impossible.

Let us indicate, to fix the ideas,

$$\begin{gathered}\vartheta_0^0 < 0, \vartheta_1^0 < 0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda+1}^0 = 0, \ldots, \\ \vartheta_\mu^0 = 0, \vartheta_{\mu+1}^0 > 0, \ldots, \vartheta_n^0 > 0\end{gathered} \tag{8}$$

where $\lambda \geq 0$ and $\mu \leq n - 2$.

Theorem: By replacing in the simplex L the vertices $(l_{ik}), k = 0, 1, 2, \ldots, \lambda$, successively by the vertex (l'_{i0}) one will obtain the simplexes

$$L_0, L_1, \ldots, L_\lambda \tag{9}$$

which are contiguous to the simplex L and one to one by the faces in $n-1$ dimensions the regulators of which are proportional to the regulator ρ_0.

Let us apply the formula (∗) of Number 74 to the simplex L and L_k $(k = 0, 1, 2, \ldots, \lambda)$; one will have

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}).$$

$$(h \neq k; n = 0, 1, 2, \ldots, \lambda)$$

By virtue of (8), one obtains,

$$\sum p_{ik}(l_{ih} - l'_{i0}) > 0. \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

In view of (8), one finds

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = \delta_k \rho_0 \text{ where } \delta_k \geq 0 \quad (k = 0, 1, 2, \ldots, \lambda)$$

and

$$\rho_k = u_k \rho_0 \text{ where } u_k > 0. \ (k = 0, 1, 2, \ldots, \lambda) \tag{10}$$

On the grounds of (1) and (7), it becomes

$$F_{(L)}(l'_{1k}, \ldots, l'_{nk}) = \omega_k F_{(L)}(l'_{10}, \ldots, l'_{n0}) \text{ where } \omega_k > 0.$$
$$(k = 0, 1, 2, \ldots, \lambda)$$

The equality obtained presents an identity with regard to the coefficients of the quadratic form $\sum\sum a_{ij}x_i x_j$. One derives, because of (2), the equalities

$$\left(\vartheta_i^{(k)}\right)^2 - \vartheta_i^{(k)} = \omega_k \left((\vartheta_i^0)^2 - \vartheta_i^0\right), \quad (i = 0, 1, 2, \ldots, n)$$
$$\vartheta_i^{(k)} \vartheta_j^{(k)} = \omega_k \vartheta_i^0 \vartheta_j^0.$$
$$(i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i \neq j)$$

As $\sum_{i=0}^{n} \vartheta_i^{(k)} = 1$ and $\sum_{i=0}^{n} \vartheta_i^{0} = 1$, it is necessary that $\omega_k = 1$ and

$$\vartheta_i^{(k)} = \vartheta_i^0, \quad (i = 0, 1, 2, \ldots, n)$$

therefore

$$l'_{ik} = l'_{i0}. \quad (k = 0, 1, 2, \ldots, \lambda)$$

The formula (1) becomes in this case

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}).$$

$$(h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

Let us notice that the simplexes

$$L, L_0, L_1, \ldots, L_\lambda$$

make up a group of perfectly determined simplexes corresponding to the independent regulator ρ_0. That which we have mentioned concerning the simplex L can be related back to all the simplexes of the series (9). All the simplexes which remain $L_{\lambda+1}, \ldots, L_n$ are contiguous to the simplex L through the faces the regulators of which are not proportional to ρ_0.

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Let us notice that the simplexes (9) make up a convex polyhedron K having $n + 2$ vertices

$$(l'_{i0}), (l_{i0}), \ldots, (l_{in}).$$

In effect, all the points of simplexes (9) belong to polyhedron K made up of points determined by the equalities

$$x_i = ul'_{i0} + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum_{u=0}^{n} u_k = 1 \text{ and } u \geq 0, u_k \geq 0. \tag{11}$$

$$(k = 0, 1, 2, \ldots, n)$$

I argue that any point (x_i) determined by these equalities belongs to at least one of simplex (9).

Let us suppose, in the first place, that one has the inequalities

$$u_k + u\vartheta_k^0 \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

One will present the equation (11), because of (2), under the following form:

$$x_i = \sum_{k=0}^{n} (u_k + u\vartheta_k^0) l_{ik}$$

and, as $\sum_{k=0}^{n}(u_k + u\vartheta_k^0) = 1$, one concludes that the point (x_i) belongs to the simplex L.

This laid down, let us suppose that at least one of numbers $u_k + u\vartheta_k^0$, $k = 0, 1, 2, \ldots, n$, is negative. Let us choose among the numbers

$$\frac{u_0}{\vartheta_0^0}, \frac{u_1}{\vartheta_1^0}, \ldots, \frac{u_\lambda}{\vartheta_\lambda^0}$$

which are all negatives or zeros, because of (8) and (11), a number $\frac{u_k}{\vartheta_\lambda^0}$ the numerical value of which is the smallest. The point (x_i) determined by the equalities (11) belongs in this case to the simplex L_k. To demonstrate this, one will present the equalities (11) under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_{i0} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) l_{ir}.$$

$$(r = 0, 1, 2, \ldots, n; r \neq k)$$

On the ground of suppositions made, one will have the inequalities

$$u + \frac{u_k}{\vartheta_k} > 0, \ u_r - u_k \frac{\vartheta_r}{\vartheta_k} \geq 0, \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

and, as

$$u + \frac{u_k}{\vartheta_k} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one concludes that the point (x_i) belongs to the simplex (L_k) $(u = 0, 1, 2, \ldots, \lambda)$.

Let us examine the faces in $n-1$ dimensions of the polyhedron K. On the ground of conditions (8), the polyhedron K possesses $\mu - \lambda$ faces in $n-1$ dimensions Q_k which are characterised by $n+1$ vertices

$$(l'_{i0}), (l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

The vertex (l_{ik}) where $k = \lambda + 1, \ldots, \mu$ is opposite to the face $\theta_k (k = \lambda + 1, \ldots, \mu)$.

All the faces in $n-1$ dimensions of the polyhedron K which remain are characterised by n vertices. One will characterise them in the polyhedron K by two opposite vertices.

One obtains in this way $n-\mu$ faces P_k $(k = \mu + 1, \ldots, n)$ of the polyhedron K characterised by two opposite vertices (l'_{i0}) and l_{ik} $(k = \mu + 1, \ldots, n)$ and one obtains $(\lambda + 1)(n - \mu)$ faces $P_{kh}(h = 0, 1, 2, \ldots, \lambda; k = \mu + 1, \ldots, n)$ characterised by two opposite vertices l_{ik} and l_{ih}.

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Let us notice that the polyhedron K is contiguous through the faces Q_k $(k = \lambda + 1, \ldots, \mu)$ to other independent regulator ρ_0.

To demonstrate this, let us examine the simplex L_k $(k = \lambda + 1, \ldots, \mu)$ contiguous to the simplex L through the face in $n-1$ dimensions characterised by the vertices

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

This face presents a part of the corresponding face Q_k of the polyhedron K.

By applying the formula (∗) of Number 74 to the simplexes L and L_k, one obtains

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0})$$

where $h \neq k$ and $n = \lambda + 1, \ldots, \mu$.

On the ground of conditions (8), one will have

$$\sum p_{ik}(l_{ih} - l'_{i0}) = 0, \quad (k = \lambda + 1, \ldots, \mu)$$

therefore, because of (7),

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}) = 2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}).$$

$$(h \neq 0; k = \lambda + 1, \ldots, \mu)$$

As the point (l'_{10}) is not found among the vertices of the simplex L_k, it is necessary that among the regulators of faces of the simplex L_k are found, by virtue of the equation obtained, regulators which are proportional to ρ_0.

By noticing that

$$l'_{i0} = \sum \vartheta^0_h l_{ih} + \vartheta^0_k l'_{ik}, \text{ where } \sum \vartheta^0_h + \vartheta^0_k = 1$$

$$(h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

since because of (8), $\vartheta^0_k = 0$ $(k = \lambda + 1, \ldots, \mu)$, one concludes, on the ground of the previous theorem, that by replacing in the simplex L_k the vertices (l_{ih}) $(h = 0, 1, 2, \ldots, \lambda)$ by the vertex (l'_{i0}) one will obtain a group of simplexes

$$L_k, L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(\lambda)}, \ (k = \lambda + 1, \ldots, \mu) \tag{12}$$

which are contiguous one to one by faces in $n-1$ dimensions the regulators of which are proportional to ρ_0.

Let us indicate by K_h the convex polyhedron made up of simplexes (12). One obtains the polyhedron K_k by replacing in the polyhedron K the vertex (l_{ik}) by the vertex l'_{ik} $(k = \lambda + 1, \ldots, \mu)$. One concludes that the polyhedra K and K_k are contiguous through the face Q_k.

Let us examine other faces of the polyhedra K. The face P_k $(k = \mu + 1, \ldots, n)$ belongs to the simplex L which is contiguous through the face P_k to the simplex L_k. The regulator ρ_k $(k = \mu + 1, \ldots, n)$ of this face can not be proportional to the independent regulator ρ_0.

It may turn out that any one of regulators corresponding to the various faces of the simplex L_k is not proportional to the regulator ρ_0. In this case, the polyhedron K will not be contiguous through the face P_k to any one analogous polyhedron corresponding to the independent regulator ρ_0.

It may also turn out that among the regulators of faces of the simplex L_k are found regulators which are proportional to ρ_0; in this case, the simplex L_k belongs to a convex polyhedron K_k which is contiguous to K through the face P_k $(k = \mu + 1, \ldots, n)$.

In the same way, one will examine the analogous faces P_{hk} of the polyhedron K $(h = 0, 1, 2, \ldots, \lambda; k = \mu + 1, \ldots, n)$.

By applying the procedure shown to the various incongruent simplexes of the set (L), one will determine the incongruent convex polyhedron

$$K, K_1, \ldots, K_{\omega-1}$$

which are made up of corresponding groups of simplexes belonging to the set (L).

Reconstruction of the set (L) of simplexes by another set (L') of simplexes.

91

One can partition the convex polyhedra

$$K, K_1, \ldots, K_{\omega-1} \tag{1}$$

corresponding to an independent regulator ρ into new simplexes.

By keeping the previous notations, let us examine the convex polyhedron K made up of simplexes

$$L, L_0, \ldots, L_\lambda. \tag{2}$$

The polyhedron K possesses $n + 2$ vertices

$$(l'_i), (l_{i0}), \ldots, (l_{in}).$$

Theorem. *By replacing in the simplex L characterised by the vertices*

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

a value (l_{ik}) *by the vertex* (l'_i) *where* $k = \mu + 1, \ldots, n$, *one obtains* $n - \mu$ *simplexes*

$$L_{\mu+1}, \ldots, L'_n \tag{3}$$

which also make up the polyhedron K. *The simplexes obtained do not belong to the set* (L) *of simplexes.*

Let us write, as we have done in Number 87,

$$l'_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \tag{4}$$

and

$$\begin{aligned}\vartheta_0 < 0, \vartheta_1 < 0, \ldots, \vartheta_\lambda < 0, \vartheta_{\lambda+1} = 0, \ldots, \\ \vartheta_\mu = 0, \vartheta_{\mu+1} > 0, \ldots, \vartheta_n > 0.\end{aligned} \tag{5}$$

It is clear that each point of simplex (3) belongs to the polyhedron K.

Let (x_i) be any one point of the polyhedron K determined with the help of equalities

$$x_i = ul'_i + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum u_k = 1, u \geq 0, u_k \geq 0. \tag{6}$$

$$(k = 0, 1, \ldots, n)$$

Let us choose among the numbers

$$\frac{u_{\mu+1}}{\vartheta_{\mu+1}}, \ldots, \frac{u_n}{\vartheta_n}$$

the one which is the smallest. Let us suppose, to fix the ideas, that

$$\frac{u_r}{\vartheta_r} \geq \frac{u_k}{\vartheta_k}. \quad (r = \mu + 1, \ldots, n)$$

I argue that the point (x_i) belongs to the simplex L'_k. In effect, the equality (6) can be speculated, because of (4), under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_i + \sum_h \left(u_h - u_k \frac{\vartheta_h}{\vartheta_k}\right) l_{ih}.$$

$$(h = 0, 1, 2, \ldots, n; h \neq k)$$

By observing that

$$u + \frac{u_k}{\vartheta_k} \geq 0, \; u_h - u_k \frac{\vartheta_h}{\vartheta_k} \geq 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

and that

$$u + \frac{u_k}{\vartheta_k} + \sum_h \left(u_k - u_k \frac{\vartheta_h}{\vartheta_k}\right) = 1 \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

one concludes that the point (x_i) belongs to the simplex L'_k $(k = \mu + 1, \ldots, n)$.

The simplexes (3) can not belong to the set (L), because this set, by virtue of Theorem I of Number 61, uniformly partition the space in n dimensions.

92

Let us suppose that one has replaced in the set (L) the group of simplexes (2) by the corresponding group (3). Let us suppose that one has effected this reconstruction of simplexes of the set (L) with regard to all the polyhedra which are congruent to the polyhedra (1). One obtains in this way a new set (L') of simplexes which enjoy the following properties.

1. The set (L') of simplexes uniformly fills the space in n dimensions.

2. The set (L') can be divided into classes of congruent simplexes and the number of different classes is finite.

Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes belonging to the set (L').

Let L' and L'_0 be any two simplexes of the set (L') which are contiguous through a face P in $n-1$ dimensions. Suppose that the two simplexes L' and L'_0 also belong to the set (L'). In this case the regulator and the characteristic of the face P in the set (L') do not change.

Let us suppose that at least one of the simplexes examined does not belong to the set (L) of simplexes. This simplex will belong in this case to a polyhedron which is congruent to a polyhedron of the series (1). Let us suppose to fix the ideas that this is the polyhedron K.

By noticing that the simplex examined is found among the simplexes (3) let us choose one of these simplexes L'_k $(k=\mu+1,\dots,n)$ and examine the regulators and the characteristics of all these faces in $n-1$ dimensions.

By virtue of the definition established, the simplex L'_k is characterised by the vertices

$$(l_{i0}),\dots,(l_{i,k-1}),(l'_i),(l_{i,k+1}),\dots,(l_{in}). \quad (k=\mu+1,\dots,n)$$

Let us indicate by P_{hk} a face in $n-1$ dimensions of the simplex L'_k which is opposite to the vertex (l_{ih}) $(h=0,1,2,\dots;h\neq k)$. By P'_k let us indicate the faces of simplex L'_k which is opposite to the vertex l'_i.

All the faces in $n-1$ dimensions of the simplex L'_k can be divided into three groups:

1. $P_{0k},P_{1k},\dots,P_{\lambda k}$ and P_k;
2. $P_{\lambda+1,k},\dots,P_{\mu k}$;
3. $P_{\mu+1,k},\dots,P_{k-1,k},P_{k+1,k},\dots,P_{nk}$.

93

Let us find the regulators of faces of the simplex L'_k belonging to the first group.

Let us examine, in the first place, the face P_k. As the face P_k is characterised by the vertices

$$(l_{i0}),\dots,(l_{i,k-1}),(l_{i,k+1}),\dots,(l_{in}),$$

it presents a face of the polyhedron K.

In the set (L) the face P_k would belong to two simplexes L and L_k. Two cases to distinguish:

First case: the simplex L_k belongs to the set (L').

Let us indicate by ρ_k the regulator and by (P_{ik}) the characteristic of the face P_k in the set (L) with regard to the simplex L.

Let us indicate by ρ'_k the regulator of the face P_k in the set (L'). The characteristics of the face P_k in the set (L') with regard to the simplex L'_k will be (P_{ik}).

By virtue of the definition established in Number 73, one can declare

$$F_{(L_k)}(l'_1, \ldots, l'_n) = 2\rho'_k \sum (-p_{ik})(l_{ih} - l'_i). \quad (h \neq k)$$

By applying the formula $(*)$ of Number 74 to the simplex L and L_k, one obtains

$$F_{(L)}(l'_1, \ldots, l'_n) = F_{(L_k)}(l'_1, \ldots, l'_n) + 2\rho_k \sum p_{ik}(l_{ih} - l'_i). \quad (7)$$

$$(h \neq k)$$

It follows that

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1, \ldots, l'_n)}{2\sum p_{ik}(-l_{ih} + l'_i)}. \quad (k = \mu + 1, \ldots, n) \qquad (8)$$

We have seen in Number 87 that the function $F_{(L)}(l'_1, \ldots, l'_n)$ is proportional to the independent regulator ρ. As, by virtue of (5),

$$\sum p_{ik}(-l_{ih} + l'_i) > 0, \quad (h \neq k; k = \lambda + 1, \ldots, \mu)$$

the formula (8) can be written

$$\rho'_k = \rho_k + \delta_k \rho \text{ where } \delta_k > 0.$$

Second case: the simplex L_k does not belong to the set (L').

In this case the simplex L_k belongs to a convex polyhedron K_k and the face P_k will belong in the set (L') to two new simplexes.

The face P_k in the set (L') belongs to the simplex L'_k and to a simplex which one obtains by replacing the vertex l'_{ik} of the simplex L_k by a new vertex which one will indicate by (l_i^0). Let us indicate by L_k^0 the simplex which one obtains by replacing in the simplex L_k the vertex l'_{ik} by the vertex (l_i^0).

By virtue of the definition established in Number 73, one will have

$$F_{(L'_k)}(l_1^0, \ldots, l_n^0) = 2\rho'_k \sum p_{ik}(l_{ih} - l_i^0). \quad (h \neq k)$$

By applying the fundamental formula $(*)$ of Number 74 to the simplexes L_k and L'_k which are contiguous through the face P_k, one obtains

$$F_{(L_k)}(l_1^0, \ldots, l_n^0) = \\ F_{(L'_k)}(l_1^0, \ldots, l_n^0) + \sum(-p_{ik})(l_{ih} - l_i^0) \tfrac{F_{(L_k)}(l'_1, \ldots, l'_n)}{\sum(-p_{ik})(l_{ih} - l'_i)}$$

and, because of (7), it becomes

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1, \ldots, l'_n)}{2\sum p_{ik}(-l_{ih} + l'_i)} + \frac{F_{(L_k)}(l_1^0, \ldots, l_n^0)}{2\sum p_{ik}(l_{ih} - l_i^0)}. \qquad (9)$$

On the ground of the supposition made, the functions

$$F_{(L)}(l'_1, \ldots, l'_n) \text{ and } F_{(L_k)}(l_1^0, \ldots, l_n^0)$$

are proportional to the independent regulator ρ. As

$$\sum p_{ik}(-l_{ih} + l'_i) > 0 \text{ and } \sum p_{ik}(l_{ih} - l_i^0) > 0,$$

$$(k = \mu + 1, \ldots, n)$$

the formula (9) can be written

$$\rho'_k = \rho_k + \delta_k \rho \text{ where } \delta_k > 0. \quad (k = \mu + 1, \ldots, n)$$

In the same way, one will examine the regulator of the face P_{hk}; $(h = 0, 1, 2, \ldots, \lambda)$.

As the face P_{hk} belongs in the set (L) to the simplex L_h $(h = 0, 1, 2, \ldots, \lambda)$, one concludes that by designating the simplex L_h with the simplex L one will return to one of the two previous cases.

94

Let us find the regulators of faces of the simplex L'_k belonging to the second group. Choose one face P_{hk} $(h = \lambda + 1, \ldots, \mu; k = \mu + 1, \ldots, n)$ in this group.

The face P_{hk} presents a part of the face Q_k of the polyhedron K.

We have indicated in Number 90 by K_h the polyhedron which is contiguous to the polyhedron K through the face Q_h. The polyhedron K_h posses $n + 1$ vertices

$$(l'_i), (l_{i0}), \ldots, (l_{i,h-1}), (l'_{ih}), (l_{i,h+1}), \ldots, (l_{in}).$$

In the set (L'), the polyhedron K_h is partitioned into simplexes

$$L'_{\mu+1,h}, \ldots, l'_{n,h}$$

which one obtains by replacing in the simplexes

$$L'_{\mu+1}, \ldots, L'_n$$

the vertex (l_{ih}) by the vertex (l'_{ih}) $(h = \lambda + 1, \ldots, \mu)$.

One concludes that the face P_{hk} belongs to the set (L') to two simplexes

$$L'_k \quad \text{and} \quad L'_{kh}.$$

Let us indicate by ρ'_{hk} the regulator corresponding to the face P_{hk} in the set (L'). Notice that the characteristic (P'_{ih}) will be the same for all the faces P_{hk} where $k = \mu + 1, \ldots, n$ because these faces make up the face Q_k of the polyhedron K.

In the set (L), the face Q_h is partitioned into faces

$$P_h, P_{h0}, \dots, P_{h\lambda}$$

of simplexes $L, L_0, \dots, L_\lambda$ which have the same characteristic (p_{ih}). One concludes that

$$p'_{ih} = p_{ih},$$

provided that the characteristic (p_{ih}) is chosen with regard to the simplexes (2).

By virtue of the definition established in Number 73, one can write

$$F_{(L_k)}(l'_{ih}, \dots, L'_{nh}) = 2\rho'_{hk} \sum p_{ih}(l_{ir} - l'_{ih}). \quad (r \neq h)$$

By applying the formula $(*)$ of Number 74 to the simplexes L and L'_k, one will have

$$F_{(L)}(l'_{ih}, \dots, L'_{nh}) = \\ F_{(L'_k)}(l'_{ih}, \dots, L'_{nh}) + \textstyle\sum p_{ik}(l_{ir} - l'_{ih}) \frac{F_{(L)}(l'_1, \dots, l'_n)}{\sum p_{ik}(l_{ir} - l'_i)}.$$

Besides, one has

$$F_{(L)}(l'_{ih}, \dots, L'_{nh}) = 2\rho_h \sum p_{ih}(l_{ir} - l'_{ih}), \quad (r \neq h)$$

and consequently

$$\rho'_{hk} = \rho_h + \frac{\sum p_{ik}(l_{ir} - l'_{ih})}{\sum p_{ih}(l_{ir} - l'_{ih})} \cdot \frac{F_{(L)}(l'_1, \dots, l'_n)}{2\sum(p_{ik} - l_{ir} + l'_i)}.$$

One can thus write

$$\rho'_{hk} = \rho_h + \delta_{hk}\rho. \quad (h = \lambda + 1, \dots, \mu; k = \mu + 1, \dots, n)$$

In the formula obtained, the number δ_{hk} can be positive, negative or zero.

95

Let us find the regulators of faces of the simplex L'_k belonging to the third group. Let P_{hk} be a face belonging to this group, $h = \mu + 1, \ldots, n$, $h \neq k$, $k = \mu + 1, \ldots, n$. The face P_{hk} belongs in the set (L') to two simplexes L'_k and L'_h of the series (3). By replacing in the simplex L'_k the vertices (l_{ih}) by the vertex (l_{ik}), one obtains the simplex L'_h. This results in that by indicating with ρ_{hk} the regulator and with $(p_i^{(hk)})$ the characteristic of the face P_{hk} with respect to the simplex L'_k, one will have

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = 2\rho'_{hk} \sum p_i^{(hk)}(l_{ir} - l_{ik}). \quad (r \neq h; r \neq k) \tag{10}$$

The equality (4) can be written

$$l_{ik} = \frac{1}{\vartheta_k} l'_i + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

By noticing that

$$\frac{1}{\vartheta_k} + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one will determine the value of the function $F_{(L'_k)}(l_{1k}, \ldots, l_{nk})$, by virtue of the formula (4) of Number 73, by the equality

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = \\ \sum\sum a_{ij} l_{ik} l_{jk} - \tfrac{1}{\vartheta_k} \sum\sum a_{ij} l'_i l'_j + \textstyle\sum_r \tfrac{\vartheta_r}{\vartheta_k} \sum\sum a_{ij} l_{ir} l_{jr}.$$
$$(r = 0, 1, 2, \ldots, n; r \neq k)$$

By recalling that because of (4)

$$F_{(L)}(l'_1, \ldots, l'_n) = \sum\sum a_{ij} l'_i l'_j - \sum_{k=0}^{n} \vartheta_k \sum\sum a_{ij} l_{ik} l_{jk},$$

and by comparing the two equalities obtained, one finds

$$F_{(L'_k)}(l_{1k}, \ldots, l_{nk}) = -\frac{1}{\vartheta_k} F_{(L)}(l'_1, \ldots, l'_n).$$

By substituting in the formula (10) the expression found of the function $F_{(L'_k)}(l_{1k}, \ldots, l_{nk})$, one obtains

$$\rho'_{hk} = -\frac{1}{\vartheta_k} \cdot \frac{F_{(L)}(l'_1, \ldots, l'_n)}{2\sum p_i^{(hk)}(l_{ir} - l_{ik})}.$$

$$(h = \mu + 1, \ldots, n; k = \mu + 1, \ldots, n; h \neq k; r \neq k)$$

One concludes that by admitting

$$\rho'_{hk} = -\delta_{hk}\rho, \quad (h = \mu + 1, \ldots, n; k = \mu + 1, \ldots, n; h \neq k)$$

one will have $\delta_{hk} > 0$.

Algorithm for the study of domains of quadratic forms which are contiguous to a given domain through the faces in $\frac{n(n+1)}{2} - 1$ *dimensions.*

96

Let us suppose that a domain Δ of quadratic form corresponding to a type of primitive parallelohedra which is characterised by the set (L) of simplexes is defined by the independent inequalities

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, m)$$

Let us suppose that one of these regulators is proportional to an independent regulator ρ and construct the set (L) of simplexes in another set (L') with the help of the procedure shown in Number 91–92.

Let us indicate by

$$\rho_1, \rho_2, \ldots, \rho_\sigma$$

all the regulators of incongruent faces of simplexes belonging to the set (L) and indicate by

$$\rho'_1, \rho'_2, \ldots, \rho'_\tau$$

all the regulators of faces of simplexes belonging to the set (L').

We have seen in Number 93–95 that all these regulators can be presented under the form

$$\begin{cases} \text{either} \quad \rho'_k = -\delta_k\rho \quad \text{where } \delta_k > 0, \\ \text{or} \quad \rho'_k = \rho_k + \delta_{kh}\rho \end{cases} \tag{1}$$

so long as a regulator ρ'_k is not proportional to ρ.

Let us examine the domain D' of quadratic forms determined by the inequalities

$$\rho'_k \geq 0. \quad (k = 1, 2, \ldots, \tau) \tag{2}$$

I argue that these inequalities define a domain of quadratic forms in $\frac{n(n+1)}{2}$ dimensions. By supposing the contrary, one will find the parameters u_k $(k = 1, 2, \ldots, \tau)$ positive or zero which reduce into an identity the equality

$$\sum_{k=1}^{\tau} u_k \rho'_k = 0 \text{ where } u_k \geq 0. \ (k = 1, 2, \ldots, \tau) \tag{3}$$

By virtue of formulae (1), this identity can be written

$$\sum_{k=1}^{\sigma} v_k \rho_k \pm v\rho = 0 \text{ where } v_k \geq 0. \ (k = 1, 2, \ldots, \sigma)$$

As the regulator ρ is independent it is necessary that $v_k = 0$ as long as a regulator ρ_k is not proportional to ρ. This results in that within the identity (3) one also has $u_k = 0$ so long as a regulator ρ'_k is not proportional to ρ. By virtue of (1), the identity (3) takes the form

$$\sum u_k(-\delta_k \rho) = 0 \text{ where } u_k \geq 0 \text{ and } \delta_k > 0,$$

which is impossible.

The domain Δ' defined by the inequalities (2) corresponds to a new type of primitive parallelohedra characterised by the set (L') of simplexes.

Let us notice that by virtue of inequalities (1), any quadratic form which is interior to the face of the domain Δ determined by the equation

$$\rho = 0 \tag{4}$$

belongs to the domain Δ' and vice-versa. One concludes that the two domain Δ and Δ' are contiguous through the face in $\frac{n(n+1)}{2}-1$ dimensions determined by the equation (4).

Set (Δ) *of domains of quadratic forms corresponding to the different types of primitive parallelohedra.*

97

With the help of the algorithm explained in the previous Number, one can determine the domains of quadratic forms

$$\Delta_1, \Delta_2, \ldots, \Delta_m \tag{1}$$

which are contiguous to the domain Δ by their faces in $\frac{n(n+1)}{2} - 1$ dimensions, then one will determine the domains which are contiguous to the domains (1) and so on.

Let us indicate by (Δ) the set composed of all the domains of quadratic terms which correspond to the various types of primitive parallelohedra.

Theorem I. *The set* (Δ) *of domains of quadratic forms uniformly divides the set of all the positive quadratic forms in* n *variables.*

Let $\varphi(x_1, x_2, \ldots, x_n)$ be an arbitrary positive quadratic form. Let us choose a form $\varphi_0(x_1, x_2, \ldots, x_n)$ which is interior to the domain Δ and let us examine a vector g made up of forms

$$f = \varphi_0 + u(\varphi - \varphi_0) \text{ where } 0 \leq u \leq 1. \tag{2}$$

By making the parameter u grow in a continuous manner in the interval $0 \leq u \leq 1$, one will determine a series of domains

$$\Delta, \Delta', \ldots, \Delta^{(k)} \tag{3}$$

which are successively contiguous through the faces in $\frac{n(n+1)}{2}-1$ dimensions and which contain the various forms of the vector g.

I argue that the series of domains (3) will always be terminated by a domain to which belong the given quadratic form φ.

To demonstrate this, let us indicate by

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\tau}) \text{ where } \tau = 2^n - 1 \tag{4}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to the domain Δ of quadratic form.

Let us indicate by the symbol $N(f)$ a sum

$$N(f) = \sum_{h=1}^{\tau} f(l_{1h}, \ldots, l_{nh})$$

of values of a form $f(x_1, x_2, \ldots, x_n)$ corresponding to the systems (4).

Let us indicate, similarly, by

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{i\tau}^{(k)}) \text{ where } \tau = 2^n - 1 \tag{5}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to a domain $\Delta^{(k)}$ of the series (3) and declare

$$N^{(k)}(f) = \sum_{h=1}^{\tau} f(l_{1h}^{(k)}, \ldots, l_{nh}^{(k)}). \quad (k = 1, 2, \ldots)$$

We have seen in Number 95 that the systems (4) and (5) are congruent with respect to the modulus 2. By virtue of the theorem of Number 48, one will have an inequality

$$N(f) < N^{(k)} f, \quad (k = 1, 2, \ldots)$$

as long as a quadratic form f is interior to the domain Δ. This results in that the inequality

$$N(f) \leq N^{(k)}(f) \quad (k = 1, 2, \ldots)$$

holds providing that a form f belongs to the domain Δ.

This stated, let us notice that by virtue of the supposition made, the form φ_0 is interior to the domain Δ, therefore one will have the inequality

$$N(\varphi_0) < N^{(k)}(\varphi_0).\ (k = 1, 2, \ldots) \tag{6}$$

Let f be a form of the vector g which belongs to the domain $\Delta^{(k)}$ of te series (3).

One will have an inequality

$$N(f) \geq N^{(k)}(f). \tag{7}$$

By noticing that because of (2)

$$N(f) = (1-u)N(\varphi_0) + uN(\varphi),$$
$$N^{(k)}(f) = (1-u)N^{(k)}(\varphi_0) + uN^{(k)}(\varphi),$$

the inequality (7) can be written

$$u\left[N(\varphi) - N^{(k)}(\varphi)\right] \geq (1-u)\left[N^{(k)}(\varphi_0) - N(\varphi_0)\right].$$

As $0 < u \leq 1$, this inequality gives, because of (6),

$$N^{(k)}(\varphi) \leq N(\varphi).$$

The quadratic form φ being positive, there exist only a limited number of different systems (5) verifying this inequality. Besides, there exist only a limited number of domains of forms belonging to the set (Δ) which are characterised by the same systems (5). One concludes this that the series (3) will always be terminated by a domain to which belong the given quadratic form φ.

Let us notice that a quadratic form φ which is interior to a domain Δ does not belong to any other domain of the set (Δ), since the primitive parallelohedron corresponding to the quadratic form φ will belong to the type characterised by the domain (Δ) and can not belong to any other type of parallelohedra.

Suppose that a positive quadratic form φ is interior to a face P in a certain number of dimensions of the domain Δ. The set of all the quadratic forms belonging to the face P will be perfectly determined by a certain type of non-primitive parallelohedra. One concludes that the form φ can not belong to the domains which are contiguous through the face P.

98

By effecting the various transformation of the set (Δ) of quadratic forms with the help of substitutions of integer coefficients and of the determinant which is equal to ± 1, one will do only the permutation of domains of the set (Δ).

One concludes that the set (Δ) of domains of forms can be divided into classes of domains composed of equivalent domains.

Theorem II. *The number of various classes of domains belonging to the set* (Δ) *is finite.*

Let us choose any one domain Δ of the set (Δ) and let φ be a form which is interior to the domain Δ. We have seen in Number 54 that the positive quadratic form can be transformed into another equivalent form φ' which enjoys the property that the system (4) corresponding to the form φ' are made up of integers which do not exceed in numerical value a fixed limit.

The form φ' is interior to a domain Δ' which is equivalent to the domain Δ.

As the domain Δ' is characterised by the systems of integers which do not exceed in numerical value a fixed limit, there exist only a limited number of identical domains in the set (Δ).

99

With the help of the algorithm introduced in Number 96, one can successively determine all the representatives

$$\Delta, \Delta_1, \ldots, \Delta_{\mu-1} \tag{8}$$

of different classes of domains belonging to the set (Δ).

The domains obtained enjoy the same property as the domains of quadratic forms which have been studied in my first mémoire cited. † It results in that the domains (8) can serve in the reduction of positive quadratic forms. By calling reduced the positive quadratic forms which belong to the domains (8), one obtains a new reduction method of positive quadratic forms which is entirely analogous to a reduction method of positive quadratic forms introduced in the cited mémoire.

On the non-primitive parallelohedra corresponding to positive quadratic forms.

100

Let us suppose that a positive quadratic form φ defines a primitive parallelohedron R.

By virtue of Theorem I of Number 97, the form φ belongs at least to the domain of the set (Δ). The form φ can not be interior to any one domain of the set (Δ) because otherwise the parallelohedron R would be primitive.

Therefore the form φ belongs to one face of domains of the set (Δ).

It results in that the coefficients of the form φ verify one of many linear equations

$$\sum\sum p_{ij} a_{ij} = 0$$

to rational coefficients p_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

One concludes that a positive quadratic form $\sum\sum a_{ij} x_i x_j$ the coefficients of which present an irreducible basis can define only one primitive parallelohedron.

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† This Journal, V. 133

Let us suppose that the examined form φ is interior to a face P to any one number of dimensions of domains belonging to the set (Δ).

Let us indicate by

$$\Delta, \Delta', \dots, \Delta^{(m)}$$

the domains of the set (Δ) which are contiguous through the face P.

By virtue of that which has been stated in Number 97, one will have the equalities

$$N(\varphi) = N'(\varphi) = \dots = N^{(m)}(\varphi).$$

One concludes that a positive quadratic form φ can belong to only a finite number of domains of the set (Δ).

101

Let us suppose that an infinite series of quadratic forms

$$f_1, f_2, \dots \qquad (1)$$

is made up of forms which are interior to the domain Δ. Suppose that the forms of this series tend towards a limit φ.

The forms (1) define an infinite series of primitive parallelohedra

$$R_1, R_2, \dots$$

belonging to the one type characterised by the domain Δ which tend towards a limit R.

One concludes that any non-primitive parallelohedron R corresponding to a positive quadratic form φ can be considered as a limit of primitive parallelohedra (2).

Let us indicate by the symbol S_ν the number of faces in ν dimensions of the non-primitive parallelohedron R and by S_ν^0 let us indicate the number of faces in ν dimensions of primitive parallelohedron (2) $(\nu = 0, 1, 2, \dots, n-1)$.

As the faces of the non-primitive parallelohedron R are made up of boundaries of faces of primitive parallelohedra belonging to the series (2), one concludes that

$$S_\nu \leq S_\nu^0. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

We have seen in Number 65 that

$$S_\nu^0 \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

and consequently

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

Principal domain of the set (Δ).

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Let us apply the general theory introduced in this mémoire to a positive quadratic form

$$f = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 \ldots - 2x_{n-1}x_n$$

where one has admitted

$$a_{11} = n \text{ and } a_{ij} = -1. \ (i \neq j; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

Let us find all the representations of the minimum of the form f in a set composed of all the systems of integers which are congruous to a system $(l_1, l_2, \ldots, l_n)$ with respect to the modulus 2. Let us admit

$$l_1 = 1, \ i = 1, 2, \ldots, \lambda \text{ and } l_i = 0, i = \lambda + 1, \ldots, n. \tag{2}$$

$$(\lambda = 1, 2, \ldots, n)$$

The problem described reduces to the study of the minimum of the form

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n)$$

in the set E composed of all the systems (x_i) of integers $x_1, x_2, \ldots, x_n$.

Let us notice that the form f, by virtue of equalities (1), can be written

$$f = \sum_{i=1}^{n} x_i^2 + \quad \sum\sum_{i<j} (x_i - x_j)^2. \tag{3}$$

Each form

$$x_i^2, \ (i = 1, 2, \ldots, n) \ \ (x_i - x_j)^2,$$
$$(i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

satisfied, by (2), the condition

$$\begin{cases} (l_i + 2x_i)^2 \geq l_i^2, & (i = 1, 2, \ldots, n) \\ (l_i - l_j + 2(x_i - x_j))^2 \geq (l_i - l_j)^2, & (i = 1, 2, \ldots, n; i < j; \\ & j = 1, 2, \ldots, n) \end{cases} \tag{4}$$

Whatever may be the integer values of $x_1, x_2, \ldots, x_n$. It follows, by (3), that

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) \geq f(l_1, l_2, \ldots, l_n).$$

For the equality

$$\varphi(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) = \varphi(l_1, l_2, \ldots, l_n)$$

to holds, it is necessary, by (3) and (4), that one had the equalities

$$(l_i + 2x_i)^2 = l_i^2, \ \ l_i - l_j + 2(x_i - x_j) = (l_i - l_j)^2.$$
$$(i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

By virtue of (2), one obtains

$$x_i = 0 \text{ or } x_i = -l_i, \quad (i = 1, 2, \ldots, n)$$

therefore the form f possesses only two representations of the minimum $(l_1, l_2, \ldots, l_n)$ and $(-l_1, -l_2, \ldots, -l_n)$ in the set examined.

By attributing to the index λ in the inequalities (2) the values $\lambda = 1, 2 \ldots, n$ and by permuting the numbers $l_1, l_2, \ldots, l_n$, one obtains $2^n - 1$ systems which characterise, by virtue of the theorem of Number 48, the faces in $n-1$ dimensions of the parallelohedron R corresponding to the positive quadratic form f.

The parallelohedron R will be defined by $2(2^n-1)$ independent inequalities

$$\begin{aligned}
&1\cdot n \pm 2x_{k_1} \geq 0,\\
&2\cdot(n-1) \pm 2(x_{k_1}+x_{k_2}) \geq 0, \quad (k_1<k_2)\\
&\qquad \dots\\
&\lambda(n-\lambda+1) \pm 2(x_{k_1}+\ldots+x_{k_\lambda}) \geq 0, \quad (k_1<k_2<\cdots<k_\lambda)\\
&\qquad \dots\\
&n\cdot 1 \pm 2(x_{k_1}+x_{k_2}+\ldots+x_{k_n}) \geq 0, \quad (k_1<k_2<\cdots<k_n)
\end{aligned}$$

where $k_1=1,2,\ldots,n$, $k_2=2,\ldots,n,\ldots,$ $k_\lambda=\lambda,\ldots,n$, $k_n=n$.

To have more convenience in the subsequent notations, let us write

$$u_0=x_1+x_2\ldots+x_n,\ u_1=-x_1,\ u_2=-x_2,\ldots,\ u_n=-x_n \tag{5}$$

and notice that all the sums

$$\pm x_{k_1},\ \pm(x_{k_1}+x_{k_2}),\ldots,\ \pm(x_{k_1}+x_{k_2}+\ldots+x_{k_n})$$

are expressed by the sums

$$u_{h_0},\ u_{h_0}+u_{h_1},\ldots,\ u_{h_0}+u_{h_1}+\ldots+u_{h_{n-1}}$$

where $h_0<h_1<h_2<\cdots<h_{n-1}$ and $h_0=0,1,2,\ldots,n$, $h_1=1,2,\ldots,n,\ldots,$ $h_{n-1}=n-1,\ n$.

The inequalities which define the parallelohedron R can be written

$$\left\{\begin{aligned}
&1\cdot n+2u_{h_0}\geq 0,\\
&2\cdot(n-1)+2(u_{h_0}+u_{h_1})\geq 0,\ (h_0<h_1)\\
&\qquad \dots\\
&n\cdot 1+2(u_{h_0}+u_{h_1}\cdots+u_{h_{n-1}})\geq 0,\ (h_0<h_1<\cdots<h_{n-1})
\end{aligned}\right. \tag{6}$$

where $h_0=0,1,2,\ldots,n$, $h_1=1,2,\ldots,n,\ldots,$ $h_{n-1}=n-1,n$.

Let us find the vertices of the parallelohedron R. To this effect, let us examine a point (α_i) verifying the equations

$$\begin{aligned} n + 2u_1 &= 0, \\ 2(n-1) + 2(u_2 + u_2) &= 0, \\ &\ldots, \\ n \cdot 1 + 2(u_1 + u_2 + \ldots + u_n) &= 0 \end{aligned} \tag{7}$$

By virtue of (5), one obtains

$$\begin{aligned} \alpha_1 &= \frac{1}{2}n, \\ \alpha_2 &= \frac{1}{2}(n-2), \\ \alpha_k &= \frac{1}{2}(n-2k+2), \\ &\ldots, \\ \alpha_n &= \frac{1}{2}(-n+2). \end{aligned} \tag{8}$$

I argue that the point obtained (α_i) presents a vertex of the parallelohedron R. To demonstrate this, let us examine the form

$$f(x_1, x_2, \ldots, x_n) + 2\sum_{i=1}^{n} \alpha_i x_i$$

or, by (8), the form

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n}(n-2i+2)x_i.$$

For the point (α_i) determined by the equalities (8) to be a vertex of the parallelohedron R, it is necessary and sufficient that the inequality

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n}(n-2i+2)x_i \geq 0 \tag{9}$$

holds in the set E. By noticing that

$$f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n}(n-2i+2)x_i =$$
$$\sum_{i=1}^{n}(x_i^2 + x_i) + \sum_{i<j}\sum \left[(x_i - x_j)^2 + x_i - x_j\right],$$

one obtains the inequalities (9) because the inequalities

$$x_i^2 + x_i \geq 0,\ (x_i - x_j)^2 + x_i - x_j \geq 0,$$
$$(i = 1, 2, \dots, n; j = 2, 3, \dots, n)$$

take place within the set E.

For the equality

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n - 2i + 2) x_i = 0 \tag{10}$$

to hold, it is necessary and sufficient that one had the equality

$$x_i^2 + x_i = 0,\ (x_i - x_j)^2 + x_i - x_j = 0.$$
$$(i = 1, 2, \dots, n; i < j; j = 2, 3, \dots, n)$$

One declares that

$$x_i = -1,\ (i = 1, 2, \dots, \lambda) \quad x_i = 0.\ (i = \lambda + 1, \dots, n)$$

By attributing to the index λ the values $\lambda = 0, 1, 2, \dots, n$ one obtains $n + 1$ systems verifying the equality (10).

$$\begin{gathered}(0, 0, \dots, 0),\\ (-1, 0, \dots, 0),\\ (-1, -1, 0, \dots, 0),\\ \dots,\\ (-1, -1, \dots, -1).\end{gathered}$$

It is thus demonstrated that the point (α_i) determined by the equations (7) presents a simple vertex of the parallelohedron R.

Let us introduce in our studies symbol

$$(h_0, h_1, h_2, \dots, h_n) \tag{11}$$

in which the indices $h_0, h_1, h_2, \dots, h_n$ present a permutation of numbers $0, 1, 2, \dots, n$ and let us agree to indicate by this symbol a point which verifies the equations

$$\begin{gathered}n + 2u_{h_0} = 0, 2(n - 1) + 2\left(u_{h_0} + u_{h_1}\right) = 0, \dots,\\ n + 2\left(u_{h_0} + u_{h_1} + \dots, u_{h_{n-1}}\right) = 0.\end{gathered} \tag{12}$$

By virtue of the definition established of the symbol (11), the vertex (α_i) of the parallelohedron R determined by the equations (7) will be characterised by the symbol

$$(1, 2, \ldots, n, 0).$$

I argue that each symbol (11) characterises a vertex of parallelohedron R.

To demonstrate this, let us effect a transformation of the parallelohedron R with the help of a substitution

$$u_1 = u'_{h_0},\ u_2 = u'_{h_1}, \ldots, u_n = u'_{h_{n-1}},\ u_0 = u'_{h_n}, \tag{13}$$

where one has admitted

$$u'_0 = x'_1 + x'_2 + \ldots + x'_n,\ u'_1 = -x'_1, \ldots,\ u'_n = -x'_n.$$

The inequalities (6) which define the parallelohedron R will be permuted by the substitution considered, therefore the parallelohedron R will be transformed into itself.

To vertex (α_i) of the parallelohedron R determined by the equations (7) will be transformed, by virtue of (13), into a vertex of the parallelohedron R determined by the equations (12), therefore the vertex will be characterised by the symbol (11).

We have demonstrated the existence of $(n+1)!$ simple vertices of the parallelohedron R corresponding to the positive quadratic form φ. As the number of vertices of any one parallelohedron corresponding to a positive quadratic form does not exceed a limit $(n+1)!$, by virtue of the formula (3) of Number 101, one concludes that the parallelohedron R does not possess vertices other than those which are characterised by the symbol

$$(h_0, h_1, \ldots, h_n)$$

in which one permutes the indices $0, 1, 2, \ldots, n$ in every possible ways.

All the vertices of the parallelohedron R are simple, therefore the parallelohedron R is primitive. By noticing that the number of vertices of the parallelohedra R is expressed by the formula

$$S_0 = (n+1)! = (n+1)\Delta^{(m)}(m^n)_{m=1}, \qquad (14)$$

one concludes, by virtue of that which has been said in Number 66, that the number S_ν of faces in ν dimensions of the parallelohedron R is expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

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Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes of the set (L) which defines the type of primitive parallelohedra to which belongs the parallelohedron R examined.

Any symbol $(h_0, h_1, \ldots, h_n)$ defines a simplex characterised by the linear functions

$$u_{h_0}, u_{h_0} + u_{h_1}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}.$$

By virtue of (5), one will have identically

$$u_{h_0} + u_{h_1} + \ldots + u_{h_n} = 0. \qquad (15)$$

Notice that $n+1$ simplexes which one obtains by carrying out the circular permutations of indices $h_0, h_1, \ldots, h_n$

$$(h_0, h_1, \ldots, h_n), (h_1, h_2, \ldots, h_0), \ldots, (h_n, h_0, \ldots, h_{n-1})$$

are congruent. By choosing a representative among these simplexes, one will determine in this manner $n!$ incongruent simplexes of the set (L).

Let us examine two simplexes determined by two symbols

$$(h_0, h_1, h_2, \ldots, h_n) \quad \text{and} \quad (h_1, h_0, h_2, \ldots, h_n)$$

which differ only by a transposition of indices h_0 and h_1.

By virtue of the definition established, these simplexes are characterised by the functions

$$[u_{h_0}, u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}] \tag{16}$$

and

$$[u_{h_1}, u_{h_1} + u_{h_0}, u_{h_1} + u_{h_0} + u_{h_2}, \ldots, u_{h_1} + u_{h_0} + \ldots + u_{h_n}] . \tag{17}$$

These two simplexes differ only by the vertices which are characterised by the function u_{h_0} and u_{h_1}.

One concludes that these two simplexes are contiguous by a face in $n-1$ dimensions which is characterised by the functions

$$[u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}] . \quad ([1]8)$$

Let us determine the characteristic $\pm(p_i)$ of this face. By declaring, as that which we have done in Number 72,

$$u^0_{h_0} + u^0_{h_1} = \delta, u^0_{h_0} + u^0_{h_1} + u^0_{h_2} = \delta, \ldots, u^0_{h_0} + \ldots, + u^0_{h_1} + u^0_{h_n} = \delta,$$

one obtains, by (15), $\delta = 0$ and consequently

$$u^0_{h_0} = -u^0_{h_1}, u^0_{h_2} = 0, \ldots, u^0_{h_n} = 0. \tag{19}$$

By indicating with (p_i) the characteristic of the face (18) with regard to the simplex (16), one will have a supplementary condition

$$u^0_{h_0} > 0 \tag{20}$$

which, added to the equalities (19), well defines the characteristic (p_i).

Let us indicate, to make short,

$$h_0 = i \quad \text{and} \quad h_1 = j \tag{21}$$

and suppose that $i \neq 0$ and $j \neq 0$. By virtue of equalities (5), one will have

$$u^0_{h_0} = u^0_i = -p_i, u^0_{h_1} = u^0_j = -p_j.$$

By virtue of (19) and (20), one obtains

$$\begin{cases} p_k = 0, \\ p_i = -1, p_j = 1. \end{cases} \quad (k = 1, 2, \ldots, n; k \neq i; k \neq j) \qquad (22)$$

One can therefore characterise the characteristic (p_i) by a corresponding function

$$\sum p_i x_i = -x_i + x_j.$$

Let us suppose that $j = 0$. One will have in this case the equalities

$$p_1 + p_2 + \ldots + p_n = p_i, p_k = 0 \quad (k = 1, 2, \ldots, n; k \neq i)$$

and consequently, by (20),

$$\sum p_i x_i = -x_i.$$

In the same way one will examine the case $i = 0$. One can bring together the three cases examined by indicating the characteristic of the face (18) by the function $-x_i + x_j$, provided that $x_0 = 0$.

Let us find the regulator ρ_{ij} of the face examined. To that effect, let us determine the number $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ after the conditions

$$u_{h_1} = \sum_{\lambda=0}^{n} \vartheta_\lambda (u_{h_0} + \ldots + u_{h_\lambda}) \text{ where } \sum \vartheta_\lambda = 1.$$

One obtains

$$\vartheta_0 = -1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1.$$

By applying the formula (20) of Number 72, one finds

$$2\rho_{ij}\left[(u^0_{h_j} + \ldots + u^0_{h_\lambda}) - u^0_{h_1}\right] = (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2 - (u_{h_0} + u_{h_1} + \ldots + u_{h_n})^2 \text{ where } \lambda > 0.$$

By virtue of equalities (15) and (19), this formula comes down to the one here

$$2\rho_{ij} = (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2,$$

and consequently

$$\rho_{ij} = -u_{h_0} u_{h_1}$$

or, by (21)

$$\rho_i = -u_i u_j. \qquad (23)$$

Let us suppose that $j = 0$; the formulae (5) give

$$\rho_{i0} = x_i(x_1 + x_2 + \ldots + x_n) = x_1 x_i + x_2 x_i + \ldots + x_n x_i.$$

By replacing in this formula $x_i x_j$ by a_{ij}, one obtains the sought-for expression of the regulator ρ_{i0}

$$\rho_{i0} = \sum_{k=1}^{n} a_{ki} \ \ (i = 1, 2, \ldots, n) \qquad (24)$$

By supposing that $i \neq 0$ and $j \neq 0$, one will have

$$\rho_{ij} = -x_i x_j$$

and consequently

$$\rho_{ij} = -a_{ij} \ \ (i = 1, 2, \ldots, \nu; i \neq j; j = 1, 2, \ldots, n) \qquad (25)$$

Observe that the face (18) possesses the regulator ρ_{ij} and the characteristic $-x_i + x_j$ in addition to values of indices $h_2, \ldots, h_n$. One concludes that there exist $(n-1)!$ incongruent faces of simplexes of the set (L) which possess the same regulator ρ_{ij} and the same characteristic

$$-x_0 + x_j. \quad (i = 0, 1, 2, \ldots, n; i \neq j; j = 0, 1, 2, \ldots, n)$$

By applying the formula (I.) from Number 84, one obtains

$$\sum\sum a_{ij} x_i x_j = \sum_{i<j}\sum \rho_{ij}(x_i - x_j)^2,$$

$$(i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

where one has admitted $x_0 = 0$, or differently

$$\sum\sum a_{ij}x_ix_j = \sum_{i=1}^{n} \rho_{i0}x_i^2 + \sum_{i<j}\sum \rho_{ij}(x_i - x_j)^2. \qquad (26)$$

$$(i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

The domain Δ of quadratic forms corresponding to the type of primitive parallelohedra examined will be determined by the inequalities

$$\rho_{ij} \geq 0 \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

or differently, according to (24) and (25), by the inequalities

$$\sum_{k=1}^{n} a_{ki} \geq 0, \ -a_{ij} \geq 0. \ (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n) \qquad (27)$$

The number of these inequalities is equal to $\frac{n(n+1)}{2}$, thus the domain of quadratic forms defined by these inequalities is a simple domain.

By attributing to the parameters ρ_{ij} $(i = 0, 1, 2, \ldots, n;$ $j = 0, 1, 2, \ldots, n; i < j)$ in the formula (26) the positive arbitrary values or zeros, one will determine all the quadratic forms belonging to the domain Δ.

One remarkable coincidence is signalling. The domain of quadratic forms (27) has been studied in my first mémoire cited † where it has been called principal domain. This domain corresponds to a principal perfect positive quadratic form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + \ldots + x_{n-1}x_n.$$

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† This Journal, V. 133

It is remarkable that the set of characteristics found

$$\pm x_i,\ \pm(x_i - x_j),\ (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

coincides with the set of representations of the minimum of the principal perfect form φ.

Domains of quadratic forms contiguous to the principal domain.

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All the faces in $\frac{n(n+1)}{2} - 1$ dimensions of the principal domain Δ are equivalent. †

It follows that all the domains of forms belonging to the set (Δ) which are contiguous to the principal domain Δ by the faces in $\frac{n(n+1)}{2} - 1$ dimensions are equivalent.

In the case $n = 2$ and $n = 3$, the set (Δ) of domains of quadratic forms is made up of a single class of domains equivalent to the principal domain.

One concludes that in the space of 2 and of 3 dimensions there is only a single type of primitive parallelohedra, provided that one does not consider as different the equivalent types which correspond to the equivalent domains of quadratic forms.

Let us suppose that $n \geq 4$ and find the domain Δ' which is contiguous to the principal domain Δ by the face determined with the help of the equation

$$\rho = -a_{12} = 0.$$

By applying the algorithm explained in Number 96, let us determine the incongruent convex polyhedra which correspond to the independent regulator ρ.

We have seen in Number 104 that the regulator $\rho = \rho_{12}$ corresponds to the common faces of simplexes defined by the symbols

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† See my mémoire cited

$$(1, 2, h_1, \ldots, h_n), (2, 1, h_2, \ldots, h_n)$$

where $h_2, h_3, \ldots, h_n$ present an arbitrary permutation of indices $0, 3, 4, \ldots, n$.

The two corresponding simplexes are characterised by the functions

$$[u_1, u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

and

$$[u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_n}] .$$

By declaring

$$u_2 = \vartheta_0 u_1 + \vartheta_1(u_1 + u_2) + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots \\ + \vartheta_n(u_1 + u_2 + \ldots + u_{h_n})$$

where $\sum_{k=0}^{n} = 1$, one obtains

$$\vartheta_0 = -1, \vartheta_1' = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1. \qquad (3)$$

As among the numbers obtained is found only one negative number ϑ_0, one concludes that the two simplexes (1) and (2) make up a polyhedron K corresponding to the independent regulator ρ.

Let us indicate by (L') the set of simplexes which characterise the domain Δ' of quadratic forms. By virtue of that which has been said in Number 91, the polyhedron K in the set (L') will be made up from simplexes which one obtains by replacing the vertices of the simplex (1) which correspond to the positive values of numbers (3) by the vertex characterised by the function u_2. As in the series (3) only two positive numbers ϑ_1 and ϑ_n are found, one obtains two simplexes characterised by the functions

$$[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}] \qquad (4)$$

and

$$[u_1, u_1 + u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2]\,. \qquad (5)$$

These two simplexes make up the polyhedron K and replace the two simplexes (1) and (2) in the set (L').

By effecting all the permutation of indices $h_2, \ldots, h_n$, one obtains $(n-1)!$ incongruent convex polyhedra which correspond to the independent regulator ρ.

By replacing in the set (L) the simplexes congruent to the simplexes (1) and (2) by the simplexes which are congruent to the simplexes (4) and (5), one will reconstruct the set (L) of simplexes into a set (L') which characterises the domain Δ'.

Notice that the number of incongruent simplexes of the set (L') is equal to $n!$ also. It follows that the number of faces in ν dimensions of primitive parallelohedra belonging to the type characterised by the domain of form Δ' is expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

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Let us find the regulators and the characteristic of faces in $n-1$ dimensions of simplexes belonging to the set (L').

Let us examine in the first place the simplexes contiguous to a face in $n-1$ dimensions which belong to the set (L) and to the set (L').

The condition necessary and sufficient for which the two simplexes characterised by the symbols

$$(h_0, h_1, h_2, \ldots, h_n) \text{ and } (h_1, h_0, h_2, \ldots, h_n), \qquad (6)$$

which are contiguous by a face in $n-1$ dimensions, also belong to the set (L'), consists in so long as within the two series

$$h_0, h_1, h_2, \ldots, h_n, h_0 \quad \text{and} \quad h_1, h_0, h_2, \ldots, h_n, h_1$$

the indices 1 and 2 are not adjacent. By declaring

$$h_0 = i \quad \text{and} \quad h_1 = j,$$

one obtains $(n-1)! - 2(n-2)!$ pairs of symbols (6) which satisfy the condition assumed.

By indicating with ρ_{ij} the regulator and with $\pm(x_i - x_j)$ the characteristic of the face common to the simplexes (6) determined in the set (L), one will have for the set (L') the same regulator

$$\rho'_{ij} = \rho_{ij}, \quad (i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

and the same characteristic $\pm(x_i - x_j)$, the regulator ρ_{12} being excluded. The regulator obtained ρ'_{ij} and the characteristic $\pm(x_i - x_j)$ belong to $(n-1)! - 2(n-2)!$ incongruent faces in $n-1$ dimensions of the set (L') of simplexes.

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This declared, let us examine the regulators and the characteristics of faces of simplexes (4) and (5) which make up the polyhedron K in the set (L').

The first group of faces of the simplex (4) is composed of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions

$$u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1, u_2, \ldots, u_{h_n}. \tag{7}$$

This face belongs in the set (L) to the simplexes characterised by the symbols

$$(2, 1, h_2, h_3, \ldots, h_n) \quad \text{and} \quad (2, h_2, 1, h_3, \ldots, h_n).$$

The second simplex also belongs to the set (L'). It follows that the simplex (4) is contiguous to the simplex $(2, h_2, 1, h_3, \ldots, h_n)$ by the face (7).

Let us declare $h_2 = i$ where $i = 0, 3, \ldots, n$ and indicate by ρ'_{i1} the regulator and by $\pm(x_1 - x_i)$ the characteristic of the face (7) in the set (L'). By applying the formula (8) of Number 93, one obtains

$$\rho'_{i1} = \rho_{i1} + \rho. \ (i = 0, 3, \ldots, n)$$

In the same manner, one will examine the regulators of the face of the simplex (4) which is opposite to the vertex u_2. By putting $h_2 = i$, one will have

$$\rho'_{i2} = \rho_{i2} + \rho. \ (i = 0, 3, \ldots, n) \tag{9}$$

Examine the first group of faces of the simplex (5). This group is made up of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions.

$$u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2.$$

This face belongs in the set (L) to the simplex $(2, 1, h_2, \ldots, h_n)$ and to the simplex congruent to the simplex
$(h_n, 1, h_2, \ldots, h_{n-1}, 2)$ and which is characterised by the functions

$$u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_{n-1}}, u_2 - u_{h_n}.$$

This simplex also belongs to the set (L'). By putting $h_n = i$ and by applying the formula (8) of Number (93), one finds

$$\rho'_{i2} = \rho_{i2} + \rho$$

The characteristic of this face will be $\pm(x_2 - x_i)$ $(i = 0, 3, \ldots, n)$.

In the same way one will examine the face of the simplex (5) which is opposite to the vertex u_2.

One will obtain by letting $h_n = i$

$$\rho'_{i1} = \rho_{i1} + \rho$$

and the characteristic will be $\pm(x_1 - x_i)$, $(i = 0, 3, \ldots, n)$.

Let us notice that the number of incongruent faces, belonging to the first group of simplexes of the set (L'), which possess the regulator determined by the formula (8) or by the formula (9), is equal to $2(n-2)!$.

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The second group of faces in $n-1$ dimensions of the simplex (4) is composed of $n-2$ faces which are opposite to the vertices

$$u_1 + u_2 + u_{h_2}, u_1 + u_2 + u_{h_2} + u_{h_3}, \ldots, u_1 + u_2 + \ldots, u_{h_{n-1}}.$$

Let us examine a face which is opposite to the vertex $u_1 + u_2 + \ldots + u_{h_k}$ $(k = 2, 3, \ldots, n-1)$.

A transposition of indices h_k and h_{k+1} in the symbol $(*)$ leads to the symbol

$$\begin{gathered}[u_1, u_2, \ldots, \\ u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}}, \\ u_1 + u_2 + \ldots + u_{h_{k+1}} + u_{h_k}, \\ \ldots, \\ u_1 + u_2 + \ldots + u_{h_n}]\end{gathered}$$

which defines a simplex belonging to the set (L') and which is contiguous to the simplex (4) by the face in $n-1$ dimensions examined.

Let us write $h_k = i, h_{k+1} = j$ and indicate by ρ'_{ij} the corresponding regulator. The corresponding characteristic will be determined by the equations

$$u_1^0 = 0, u_2^0 = 0, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{k-1}}^0 = 0,$$

$$u_1^0 + u_2^0 + \ldots + u_{h_{k+1}}^0 = 0, u_1^0 + u_2^0 + \ldots + u_{h_n}^0 = 0.$$

One obtains

$$u_i^0 = -u_j^0 \text{ and } u_r^0 = 0. \quad (r = 0, 1, 2, \ldots, n; r \neq i; r \neq j)$$

It follows that the characteristic will be represented by the function $\pm(x_i - x_j)$.

By declaring

$$\begin{aligned} u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}} = \\ \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots \\ +\vartheta_k(u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_k}) + \ldots \\ +\vartheta_n(u_1 + u_2 + \ldots + u_{h_n}) \end{aligned} \tag{10}$$

where $\sum_{r=0}^{n} \vartheta_r = 1$, one obtains

$$\begin{gathered} \vartheta_0 = 0, \vartheta_1 = 0, \ldots, \\ \vartheta_{k-2} = 0, \vartheta_{k-1} = 1, \vartheta_k = -1, \vartheta_{k+1} = 1, \vartheta_{k+2} = 0, \ldots, \\ \vartheta_n = 0, \end{gathered}$$

provided that $k > 2$.

The regulator ρ'_{ij} will be determined by the formula

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}})^2 \\ & - (u_1 + u_2 + \ldots + u_{h_{k-1}})^2 + (u_1 + u_2 + \ldots + u_{h_k})^2 \\ & - (u_1 + u_2 + \ldots + u_{h_{k+1}})^2. \end{aligned}$$

After the reductions, one finds

$$\rho'_{ij} = -u_{h_k} u_{h_{k+1}}$$

or differently

$$\rho'_{ij} = -u_i u_j,$$

thus, by virtue of the formula (23) of Number 104, one will have

$$\rho'_{ij} = \rho_{ij}. \; (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n; k = 3, 4, \ldots, n-1) \tag{11}$$

Let us examine the case $k = 2$. The equality (10) gives in this case

$$\begin{gathered} \vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = -1, \vartheta_3 = 1, \vartheta_4 = 0, \ldots, \\ \vartheta_{n-1} = 0, \vartheta_n = -1, \end{gathered}$$

and consequently

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + u_{h_3})^2 - u_1^2 - u_2^2 + (u_1 + u_2 + u_{h_2})^2 \\ & - (u_1 + u_2 + u_{h_2} + u_{h_3})^2 + (u_1 + u_2 + \ldots + u_{h_n})^2. \end{aligned}$$

As $u_{h_2} = i$, $u_{h_3} = j$ and $u_1 + u_2 + \ldots + u_{h_n} = 0$, one obtains

$$\rho'_{ij} = u_1 u_2 - u_i u_j,$$

thus one will have in the case examined

$$\rho'_{ij} = \rho_{ij} - \rho. \ (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n) \qquad (12)$$

In the same way, one will examine the faces of the simplex (5) belonging to the second group and one will obtain the same formulae (11) and (12).

Let us notice that the number of incongruent faces of simplexes of the set (L') which belong to the second group and the regulator of which is determined by the formula (11) is equal to $(n-3)!2(n-3)$. The number of regulators which are determined by the formula (12) is equal to $2(n-3)!$.

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The third group of faces of simplexes (4) and (5) is composed of a single face

$$[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}]$$

which is common to these two simplexes.

The characteristic of this face is determined by the equations

$$u_1^0 = \delta, u_2^0 = \delta, u_1^0 + u_2^0 + u_{h_2}^0 = \delta, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{n-1}}^0 = \delta.$$

It results in that

$$u_1^0 = u_2^0 = \delta, u_2^0 = -\delta, u_{h_3}^0 = 0, \ldots, u_{h_{n-1}}^0 = 0, u_{h_n}^0 = \delta.$$

One concludes that $\delta = \pm 1$. By admitting

$$h_2 = i \quad \text{and} \quad h_n = j,$$

one obtains the characteristic $\pm(x_1 + x_2 - x_i - x_j)$.

Let us indicate the corresponding regulator by ρ'_{ij}. with the help of equalities

$$\begin{aligned} u + u_2 =& \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2 (u_1 + u_2 + u_{h_2}) + \dots \\ &+ \vartheta_n (u_1 + u_2 + \dots + u_{h_n}) \text{ where } \sum_{k=1}^{n} \vartheta_k = 1, \end{aligned}$$

one obtains

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = 0, \dots, \vartheta_{n-1} = 0, \vartheta_n = -1.$$

The regulator ρ'_{ij} will be determined by the formula

$$2\rho'_{ij} = (u_1 + u_2)^2 - u_1^2 - u_2^2 - (u_1 + u_2 + \dots + u_{h_n}),$$

and it becomes

$$\rho'_{ij} = u_1 u_2,$$

thus

$$\rho'_{ij} = -\rho. \quad (i = 0, 3, \dots, n; j = 0, 3, \dots, n)$$

The number of incongruent faces belonging to the third group having the characteristic $\pm(x_1 + x_2 - x_i - x_j)$ is equal to $2(n-3)!$.

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With the help of deduced formulae, one can determine all the independent regulators. Let us admit

$$\rho'_{12} = -\rho, \ \rho'_{i1} = \rho i1 + \rho, \ \rho'_{i2} = \rho i2 + \rho, \ \rho'_{ij} = \rho_{ij}. \qquad (13)$$

$$(i = 0, 3, \dots, n; i < j; j = 3, \dots, n)$$

One obtains in this manner $\frac{n(n+1)}{2}$ independent regulators ρ'_{ij} $(i = 0, 1, 2, \dots, n; i < j; j = 1, 2, \dots, n)$.

The results of our studies can be gathered in the following table:

1. Regulator ρ'_{12}, characteristic $\pm(x_1+x_2-x_i-x_j)$, their number $2(n-3)!$
2. Regulator ρ'_{i1}, characteristic $\pm(x_1-x_i)$, their number $2(n-2)!$
3. Regulator ρ'_{i2}, characteristic $\pm(x_2-x_i)$, their number $2(n-2)!$
4. Regulator $\rho'_{i1}+\rho'_{12}$, characteristic $\pm(x_1-x_i)$, their number $(n-1)!-2(n-2)!$
5. Regulator $\rho'_{i2}+\rho'_{12}$, characteristic $\pm(x_2-x_i)$, their number $(n-1)!-2(n-2)!$
6. Regulator ρ'_{ij}, characteristic $\pm(x_i-x_j)$, their number $(n-1)!-2(n-3)!$
7. Regulator $\rho'_{ij}+\rho'_{12}$, characteristic $\pm(x_i-x_j)$, their number $2(n-3)!$

The indices i and j are the values $0, 3, \ldots, n$ and one has admitted $x_0 = 0$.

The domain Δ' of quadratic forms corresponding to the set (L') of simplexes is determined by $\frac{n(n+1)}{2}$ independent inequalities

$$\rho'_{ij} \geq 0. \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i \neq j)$$

As a result, the domain Δ' is simple.

By applying the formula (I) of Number 84, one will determine, by (13), any quadratic form $\sum\sum a_{ij}x_ix_j$ by the following formula

$$\sum\sum a_{ij}x_ix_j = \sum_{i<j}\sum \rho'_{ij}(x_i - x_j)^2 + \rho'_{12}\omega, \tag{14}$$

$$(i = 0, 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where one has admitted $x_0 = 0$ and

$$\begin{aligned} \omega = &(n-2)x_1^2 + (n-2)x_2^2 + 2x_3^2 + \ldots \\ &+ 2x_n^2 + 2x_1x_2 + 2x_1x_3 - \ldots \\ &- 2x_1x_n - 2x_2x_3 - \ldots - 2x_2x_n. \end{aligned}$$

By attributing to the independent parameters ρ'_{ij} $(i = 0, 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, n)$ all the positive values or null, one will determine by the formula (14) all the quadratic forms belonging to the domain Δ'.

One coincidence is to be pointed out: the domain Δ' presents a part of the domain R, corresponding to the perfect form φ, which has been determined in my mémoire cited. The set composed of linear forms

$$\pm(x_i - x_j) \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

(the form $\pm(x_1 - x_2)$ being excluded) and of forms

$$(x_1 + x_2 - x_i - x_j) \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n)$$

where one has put $x_0 = 0$, coincides with the linear forms which define all the representations of the minimum of the perfect form φ_1.

Parallelohedra in two dimensions

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The set (Δ) of domains of binary quadratic forms is composed of a single class of domains which are equivalent to the principal domain Δ determined by the inequalities

$$a + b \geq 0, \quad -b \geq 0, \quad c + b \geq 0.$$

Here are the conditions of reduction of binary positive quadratic forms $ax^2 + 2bxy + cy^2$ due to Selling. ‡ Any quadratic form belonging to the principal domain Δ can be determined by the equalities

$$ax^2 + 2bxy + cy^2 = \lambda x^2 + \mu y^2 + \nu(x - y)^2$$

where

$$\lambda > 0, \ \mu \geq 0, \ \nu \geq 0.$$

The parameters λ, μ and ν present the regulators of the hexagon of Lejeune Dirichlet ‡ defined by the inequalities

$$\begin{aligned} -\frac{1}{2}(\lambda + \nu) \leq x \leq \frac{1}{2}(\lambda + \nu), \\ -\frac{1}{2}(\mu + \nu) \leq y \leq \frac{1}{2}(\mu + \nu), \qquad . \\ -\frac{1}{2}(\lambda + \mu) \leq x + y \leq \frac{1}{2}(\lambda + \mu), \end{aligned}$$

By attributing to the arbitrary parameters λ, μ, ν the positive values, one will determine by these inequalities a primitive parallelohedron in two dimensions, that is to say a hexagon of Lejeune Dirichlet.

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‡ See the Introduction

By nullifying one of the parameters λ, μ, ν, for example ν, one will obtain four independent inequalities.

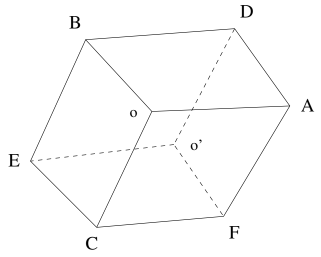
$$-\frac{1}{2}\lambda \leq x \leq \frac{1}{2}\lambda,$$
$$-\frac{1}{2}\mu \leq y \leq \frac{1}{2}\mu,$$

which define a non-primitive parallelohedron in two dimensions, it is a parallelogram.

It is easy to demonstrate that other non-primitives of the space in two dimensions do not exist.

Each hexagon of Lejeune Dirichlet can be constructed from three parallelograms as is indicated in Fig. 1.

Fig. 1



One of the three parallelograms $OADB$, $OBEC$ and $OCFA$ which form the hexagon $ADBECF$ can be arbitrarily chosen. By choosing, for example, an arbitrary parallelogram $OADB$, one will determine the two remaining parallelograms $OBEC$ and $OCFA$ by taking an arbitrary vector OC, provided that by extending this vector in the inverse direction one passes through the chosen parallelogram $OADB$.

Observe that in general the point O does not present the centre of the hexagon $ADBECF$.

One can make up the same hexagon of three parallelograms $O'DBE$, $O'ECF$ and $O'FAD$. One concludes that the hexagon of Lejeune Dirichlet does not present anything other than projection of a parallelepiped on the plane.

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The set (Δ) of domains of ternary quadratic forms is composed of a single class of domains equivalent to the principal domain Δ determined by the inequalities

$$\begin{gathered} a + b' + b'' \geq 0, \\ a' + b'' + b \geq 0, \\ a'' + b + b' \geq 0, \\ -b \geq 0, \quad -b' \geq 0, \quad -b'' \geq 0. \end{gathered}$$

Here are the conditions of reduction of ternary positive quadratic forms $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ due to Selling.

Any ternary quadratic form belonging to the principal domain Δ can be determined by the equalities

$$\begin{aligned} & ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \\ & \quad \lambda x^2 + \lambda' y^2 + \lambda'' z^2 + \mu(y-z)^2 + \mu'(z-x)^2 + \mu''(x-y)^2. \end{aligned}$$

All the primitive parallelohedra in three dimensions can be transformed with the help of linear substitutions into primitive parallelohedra determined by 14 independent inequalities

$$\left\{\begin{array}{lrcccl}
(1') & -\frac{1}{2}(\lambda+\mu'+\mu'') & \leq & x & \leq & \frac{1}{2}(\lambda+\mu'+\mu''), \quad (1'),\\
(2') & -\frac{1}{2}(\lambda'+\mu''+\mu) & \leq & y & \leq & \frac{1}{2}(\lambda'+\mu''+\mu), \quad (2'),\\
(3') & -\frac{1}{2}(\lambda''+\mu+\mu') & \leq & z & \leq & \frac{1}{2}(\lambda''+\mu+\mu'), \quad (3'),\\
(4') & -\frac{1}{2}(\lambda'+\lambda''+\mu'+\mu'') & \leq & y+z & \leq & \frac{1}{2}(\lambda'+\lambda''+\mu'+\mu''), \quad (4'),\\
(5') & -\frac{1}{2}(\lambda_{''}+\lambda+\mu_{''}+\mu) & \leq & z+x & \leq & \frac{1}{2}(\lambda_{''}+\lambda+\mu_{''}+\mu), \quad (5'),\\
(6') & -\frac{1}{2}(\lambda+\lambda_{'}+\mu+\mu') & \leq & x+y & \leq & \frac{1}{2}(\lambda+\lambda_{'}+\mu+\mu'), \quad (6'),\\
(7') & -\frac{1}{2}(\lambda+\lambda'+\lambda'') & \leq & x+y+z & \leq & \frac{1}{2}(\lambda+\lambda'+\lambda''), \quad (7')
\end{array}\right. \qquad (2)$$

The parameter $\lambda, \lambda_{\prime}, \lambda_{\prime\prime}, \mu, \mu_{\prime}, \mu_{\prime\prime}$ present the independent regulators of the primitive parallelohedron defined by these inequalities and corresponding to a ternary positive quadratic form (1).

By virtue of the formula (14) of Number 103, any primitive parallelohedron of the space in three dimensions possesses 24 vertices which can be characterised by three numbers corresponding to the different faces in two dimensions of the parallelohedron defined by the inequalities (2).

One will divide these (24) vertices into three groups I, II and III:

I	$1\ 6\ 7$	$1'\ 2\ 4$	$2'\ 6'\ 3$	$4'\ 3'\ 7'$
	$1'\ 6'\ 7'$	$1\ 2'\ 4'$	$2\ 6\ 3'$	$4\ 3\ 7$
II	$1\ 7\ 5$	$1'\ 4\ 3$	$4'\ 7'\ 2'$	$3'\ 2\ 5'$
	$1'\ 7'\ 5'$	$1\ 4'\ 3'$	$4\ 7\ 2$	$3\ 2'\ 5$
III	$1\ 3'\ 6$	$1'\ 5'\ 2$	$5\ 3\ 7$	$2'\ 7'\ 6'$
	$1'\ 3\ 6'$	$1\ 5\ 2'$	$5'\ 3'\ 7'$	$2\ 7\ 6$

Each line of this table is composed of four congruent vertices. In each group the second line is formed from vertices opposite to those which are found in the first line.

Let us examine the regulators and the characteristics of edges of the primitive parallelohedron in three dimensions. It suffices to examine the regulators and the characteristics of faces in two dimensions belonging to three simplexes

$$(0, 1, 6, 7), \ (0, 1, 7, 5), \ (0, 1, 3', 6).$$

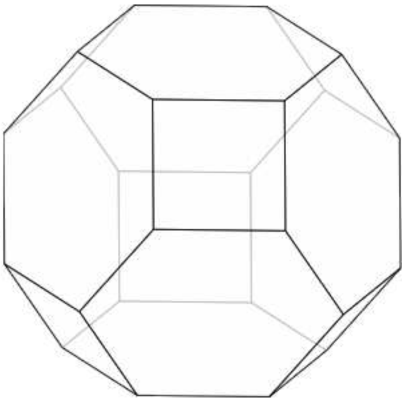
The results of these studies can be brought together in the following table:

I	0 1 6 7	-1 0 0 2 1 1	λ	1 -1 0 0 1 0	μ''	0 1 -1 1 0 1	μ	0 0 1 0 0 -1	λ''
II	0 1 7 5	-1 0 0 2 1 1	λ	1 0 -1 0 0 1	μ'	0 1 0 0 -1 0	λ'	0 -1 1 1 1 0	μ
III	0 1 $3'$ 6	-1 0 1 1 0 -1	μ'	1 -1 0 0 1 9	μ''	0 0 -1 1 1 1	$\lambda_{\prime\prime}$	0 1 0 0 -1 -1	$\lambda_{\prime}$

The first line contains the characteristics and the regulators of different faces of the corresponding simplexes. The second line is composed of vertices which define the simplexes contiguous to the corresponding simplex by the faces the characteristics of which are found above, in the first line.

The faces of primitive parallelohedron R in three dimensions (Fig. 2) are divided into 8 hexagons of Lejeune Dirichlet and into 6 parallelograms.

Fig. 2



The hexagonal faces of the primitive parallelohedron R are characterised by the numbers

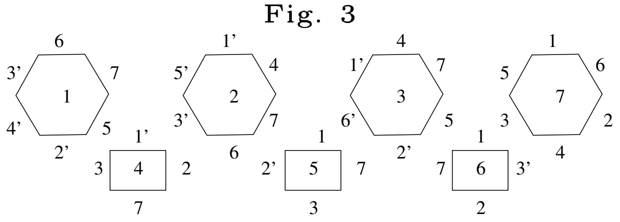
$$1, 2, 3, 7, 1', 2', 3', 7'.$$

The parallelogrammatic faces are characterised by the numbers

$$4, 5, 6, 4', 5', 6'.$$

The faces, the edges and the vertices of the primitive parallelohedron are systematically characterised in Fig. 3.

Fig. 3



One has indicated in this figure the numbers of faces which are contiguous to one of 7 incongruent faces. Each edge is characterised by two adjacent numbers and each vertex by three numbers.

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By nullifying one or more parameters $\lambda, \lambda', \lambda'', \mu, \mu', \mu''$ in the inequalities (2), one will determine the non-primitive parallelohedra in three dimensions. It is easy to see that the non-primitive parallelohedra obtained are divided into four different spaces. (Fig. 4–7)

Fig. 4

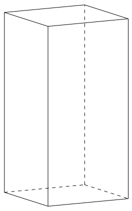


Fig. 5

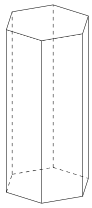


Fig. 6

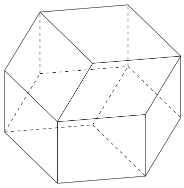
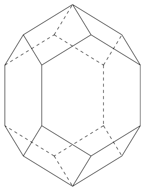


Fig. 7



Non-primitive parallelohedra of the 1^{st} *space.* By making $\mu = 0, \mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 6 independent inequalities

$$\begin{array}{rcccl} -\frac{1}{2}\lambda & \leq & x & \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}\lambda' & \leq & y & \leq & \frac{1}{2}\lambda', \\ -\frac{1}{2}\lambda'' & \leq & z & \leq & \frac{1}{2}\lambda'', \end{array}$$

which define a parallelepiped (Fig. 4).

Non-primitive parallelohedra of the 2^{nd} *space.* By making $\mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 8 independent inequalities

$$\begin{array}{rcccl} -\frac{1}{2}\lambda & \leq & x & \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}(\lambda' + \mu) & \leq & y & \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\lambda'' + \mu) & \leq & z & \leq & \frac{1}{2}(\lambda'' + \mu), \\ -\frac{1}{2}(\lambda' + \lambda'') & \leq & y + z & \leq & \frac{1}{2}(\lambda' + \lambda''), \end{array}$$

which define a prism with hexagonal base (Fig. 5).

Non-primitive parallelohedra of the 3^{rd} *space.* By making $\lambda'' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcccl} -\frac{1}{2}(\lambda + \mu') & \leq & x & \leq & \frac{1}{2}(\lambda + \mu'), \\ -\frac{1}{2}(\lambda' + \mu) & \leq & y & \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\mu + \mu') & \leq & z & \leq & \frac{1}{2}(\mu + \mu'), \\ -\frac{1}{2}(\lambda' + \mu') & \leq & y + z & \leq & \frac{1}{2}(\lambda' + \mu'), \\ -\frac{1}{2}(\lambda + \mu) & \leq & z + x & \leq & \frac{1}{2}(\lambda + \mu), \\ -\frac{1}{2}(\lambda + \lambda') & \leq & x + y + z & \leq & \frac{1}{2}(\lambda + \lambda'), \end{array}$$

which define a parallelogrammatic dodecahedron (Fig. 6).

Non-primitive parallelohedra of the 4^{th} *space.* By making $\mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcccl} -\frac{1}{2}(\lambda + \mu') & \leq & x & \leq & \frac{1}{2}(\lambda + \mu'), \\ -\frac{1}{2}(\lambda' + \mu) & \leq & y & \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\lambda'' + \mu + \mu') & \leq & z & \leq & \frac{1}{2}(\lambda'' + \mu + \mu'), \\ -\frac{1}{2}(\lambda' + \lambda'' + \mu') & \leq & y + z & \leq & \frac{1}{2}(\lambda' + \lambda'' + \mu'), \\ -\frac{1}{2}(\lambda'' + \lambda + \mu) & \leq & z + x & \leq & \frac{1}{2}(\lambda'' + \lambda + \mu), \\ -\frac{1}{2}(\lambda + \lambda' + \lambda'') & \leq & x + y + z & \leq & \frac{1}{2}(\lambda + \lambda' + \lambda''), \end{array}$$

which define a dodecahedron in 4 hexagonal faces and in 8 parallelogrammatic face (Fig. 7). Mr. Fedorow has demonstrated that other parallelohedra in three dimensions do not exist. ¶

Parallelohedra in four dimensions

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¶ See the Introduction

The first type of primitive parallelohedra in four dimensions is characterised by the principal domain Δ of quaternary quadratic forms which is determined by the independent inequalities

$$\begin{aligned}
\lambda_1 &= a_{11}+a_{12}+a_{13}+a_{14} \geq 0,\\
\lambda_2 &= a_{21}+a_{22}+a_{23}+a_{24} \geq 0,\\
\lambda_3 &= a_{31}+a_{32}+a_{33}+a_{34} \geq 0,\\
\lambda_4 &= a_{41}+a_{42}+a_{43}+a_{44} \geq 0,\\
\mu_1 &= -a_{12} \geq 0,\\
\mu_2 &= -a_{13} \geq 0,\\
\mu_3 &= -a_{14} \geq 0,\\
\mu_4 &= -a_{23} \geq 0,\\
\mu_5 &= -a_{24} \geq 0,\\
\mu_6 &= -a_{34} \geq 0.
\end{aligned}$$

Any quaternary quadratic form

$$f(x_1, x_2, x_3, x_4)$$

belonging to the domain Δ can be determined by the equalities

$$\begin{aligned}
f(x_1,x_2,x_3,x_4) = {}& \lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2+\lambda_4 x_4^2+\mu_1(x_1-x_2)^2+\\
&\mu_2(x_1-x_3)^2+\mu_3(x_1-x_4)^2+\mu_4(x_2-x_3)^2+\\
&\mu_5(x_2-x_4)^2+\mu_6(x_3-x_4)^2.
\end{aligned} \tag{1}$$

The corresponding parallelohedron is determined by 30 inequalities which one will write down in the form

$$\pm(l_1x_1+l_2x_2+l_3x_3+l_4x_4) \leq \frac{1}{2} f(l_1,l_2,l_3,l_4). \tag{2}$$

The systems $\pm(l_1, l_2, l_3, l_4)$ and the corresponding values

$$f(l_1, l_2, l_3, l_4)$$

of the quadratic form (1) are given in the following table:

Ist *type of parallelohedra*

N	l_1	l_2	l_3	l_4	$f(l_1, l_2, l_3, l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	$2'$
1	1	0	0	0	$\lambda_1 + \mu_1 + \mu_2 + \mu_3$	-1	0	0	0	$1'$
2	0	2	0	0	$\lambda_2 + \mu_1 + \mu_4 + \mu_5$	0	-1	0	0	$2'$
3	0	0	1	0	$\lambda_3 + \mu_2 + \mu_4 + \mu_6$	0	0	-1	0	$3'$
4	0	0	0	1	$\lambda_4 + \mu_3 + \mu_5 + \mu_6$	0	0	0	-1	$4'$
5	1	1	0	0	$\lambda_1 + \lambda_2 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	-1	-1	0	0	$5'$
6	1	0	1	0	$\lambda_1 + \lambda_3 + \mu_1 + \mu_3 + \mu_4 + \mu_6$	-1	0	-1	0	$6'$
7	1	0	0	1	$\lambda_1 + \lambda_4 + \mu_1 + \mu_2 + \mu_5 + \mu_6$	-1	0	0	-1	$7'$
8	0	1	1	0	$\lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_5 + \mu_6$	0	-1	-1	0	$8'$
9	0	1	0	1	$\lambda_2 + \lambda_4 + \mu_1 + \mu_3 + \mu_4 + \mu_6$	0	-1	0	-1	$9'$
10	0	0	1	1	$\lambda_3 + \lambda_4 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0	0	-1	-1	$10'$
11	1	1	1	0	$\lambda_1 + \lambda_2 + \lambda_3 + \mu_3 + \mu_5 + \mu_6$	-1	-1	-1	0	$11'$
12	1	1	0	1	$\lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_4 + \mu_6$	-1	-1	0	-1	$12'$
13	1	0	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_4 + \mu_5$	-1	0	-1	-1	$13'$
14	0	1	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3$	0	-1	-1	-1	$14'$
15	1	1	1	1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$	-1	-1	-1	-1	$15'$

By attributing to the parameters

$$\lambda_1,\ \lambda_2, \ldots,\ \mu_6$$

the arbitrary positive values, one will determine with the help of inequalities (2) all the primitive parallelohedra of the first type.

The primitive parallelohedra of the first type possess 120 vertices which can be divided into 12 groups composed of congruent vertices and of opposite vertices.

All these vertices are put together in the following table:

Vertices of primitive parallelohedron of Ist *type*

I	1	5	11	15	1′	2	8	14	2′	5′	3	10	8′	3′	11′	4	14′	10′	4′	15′
	1′	5′	11′	15′	1	2′	8′	4′	2	5	3′	10′	8	3	11	4′	14	10	4	15
II	1	5	3′	10′	1′	2	6′	13′	2′	5′	11′	15′	6	11	3	4′	13	15	4	10
	1′	5′	3	10	1	2′	6	13	2	5	11	15	6′	11′	3′	4	13′	15′	4′	10′
III	1	6	11	4′	1′	3	8	7′	3′	6′	2	13′	8′	2′	11′	15′	7	13	15	4
	1′	6′	11′	4	1	3′	8′	7	3	6	2′	13	8	2	11	15	7′	13′	15′	4′
IV	1	7	13	15	1′	4	10	14	4′	7′	3	8	10′	3′	13′	2	14′	8′	2′	15′
	1′	7′	13′	15′	1	4′	10′	14′	4	7	3′	8′	10	3	13	2′	14	8	2	15
V	1	5	12	15	1′	2	9	14	2′	5′	4	10	9′	4′	12′	3	14′	10′	3′	15′
	1′	5′	12′	15′	1	2′	9′	14′	2	5	4′	10′	9	4	12	3′	14	10	3	15
VI	1	5	4′	10′	1′	2	7′	13′	2′	5′	12′	15′	7	12	4	3′	13	15	3	10
	1′	5′	4	10	1	2′	7	13	2	5	12	15	7′	12′	4′	3	13′	15′	3′	10′
VII	1	7	12	3′	1′	4	9	6′	4′	7′	2	13′	9′	2′	12′	15′	6	13	15	3
	1′	7′	12′	3	1	4′	9′	6	4	7	2′	13	9	2	12	15	6′	13′	15′	3′
VIII	1	6	13	15	1′	3	10	14	3′	6′	4	9	10′	4′	13′	2	14′	9′	2′	15′
	1′	6′	13′	15′	1	3′	10′	14′	3	6	4′	9′	10	4	13	2′	14	9	2	15
IX	1	5	11	4′	1′	2	8	7′	2′	5′	3	12′	8′	3′	11′	15′	7	12	15	4
	1′	5′	11′	4′	1	2′	8′	7	2	5	3′	12	8	3	11	15	7′	12′	15′	4′
X	1	5	12	3′	1′	2	9	6′	2′	5′	4	11′	9′	4′	12′	15′	6	11	15	3
	1′	5′	12′	3	1	2′	9′	6	2	5	4′	11′	9	4	12	15	6′	11′	15′	3′
XI	1	6	11	15	1′	3	8	14	3′	6′	2	9	8′	2′	11′	4	14′	9′	4′	15′
	1′	6′	11′	15′	1	3′	8′	14′	3	6	2′	9′	8	2	11	4′	14	9	4	15
XII	1	7	12	15	1′	4	9	14	4′	7′	2	8	9′	2′	12′	3	14′	8′	3′	15′
	1′	7′	12′	15′	1	4′	9′	14′	4	7	2′	8′	9	2	12	3′	14	8	3	15

Regulators and characteristics corresponding to the Ist *type of parallelohedra.*

I	0 1 5 11 15	−1 0 0 0	λ_1	1 −1 0 0	0 1 −1 0	0 0 1 −1	0 0 0 1
		2 1 1 1		(2) μ_1	(6) μ_4	(12) μ_6	(4′) λ_4
II	0 1 5 3′ 10′	−1 0 1 0	μ_2	1 −1 0 0	0 1 0 0	0 0 −1 1	0 0 0 1
		1 0 −1 0		(2) μ_1	(14′) λ_2	(4′) μ_6	(12) λ_4
III	0 1 6 11 4′	−1 0 0 1	μ_3	1 0 −1 0	0 −1 1 0	0 1 0 0	0 0 0 −1
		1 0 0 −1		(3) μ_2	(5) μ_4	(9′) λ_2	(15) λ_4
IV	0 1 7 13 15	−1 0 0 0	λ_1	1 0 0 −1	0 0 −1 1	0 −1 1 0	0 1 0 0
		2 1 1 1		(4) μ_3	(6) μ_6	(12) μ_4	(2′) λ_2
V	0 1 5 12 15	−1 0 0 0	λ_1	1 −1 0 0	0 1 0 −1	0 0 −1 1	0 0 1 0
		2 1 1 1		(2) μ_1	(7) μ_5	(11) μ_6	(3′) λ_3
VI	0 1 5 4′ 10′	−1 0 0 1	μ_3	1 −1 0 0	0 1 0 0	0 0 1 −1	0 0 −1 0
		1 0 0 −1		(2) μ_1	(14′) λ_2	(3′) μ_6	(11) λ_3
VII	0 1 7 12 3′	−1 0 1 0	μ_2	1 0 0 −1	0 −1 0 1	0 1 0 0	0 0 −1 0
		1 0 −1 0		(4) μ_3	(5) μ_5	(8′) λ_2	(15) λ_3
VIII	0 1 6 13 15	−1 0 0 0	λ_1	1 0 −1 0	0 0 1 −1	0 −1 0 1	0 1 0 0
		2 1 1 1		(3) μ_2	(7) μ_6	(11) μ_5	(2′) λ_2
IX	0 1 5 11 4′	−1 0 0 1	μ_3	1 −1 0 0	0 1 −1 0	0 0 1 0	0 0 0 −1
		1 0 0 −1		(2) μ_1	(6) μ_4	(10′) λ_3	(15) λ_4
X	0 1 5 12 3′	−1 0 1 0	μ_2	1 −1 0 0	0 1 0 −1	0 0 0 1	0 0 −1 0
		1 0 −1 0		(2) μ_1	(7) μ_5	(10′) λ_4	(15) λ_3
XI	0 1 6 11 15	−1 0 0 0	λ_1	1 0 −1 0	0 −1 1 0	0 1 0 −1	0 0 0 1
		2 1 1 1		(3) μ_2	(5) μ_4	(13) μ_5	(4′) λ_4
XII	0 1 7 12 15	−1 0 0 0	λ_1	1 0 0 −1	0 −1 0 1	0 1 −1 0	0 0 1 0
		2 1 1 1		(4) μ_3	(5) μ_5	(13) μ_4	(3′) λ_3

In this table, the first line of each group contains the characteristics of faces in three dimensions corresponding to the simplexes $I, II, \ldots, XII$.

The second line contains the vertices of simplexes which are contiguous to the simplexes $I, II, \ldots, XII$ by the faces, the characteristic of which are indicated above in the first line, and the regulators are indicated near by in the second line.

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Let us examine the parallelohedra in four dimensions which belong to the second type of primitive parallelohedra defined by the domain Δ' of quaternary quadratic forms. The domain Δ' is contiguous to the principal domain Δ by the face in a dimensions defined by the equation

$$\mu_1 = 0.$$

The independent regulator μ_1 corresponds to the faces of simplexes

$$I,\ II,\ V,\ VI,\ IX,\ ,X.$$

All these simplexes have to be reconstructed with the help of the algorithm explained in Number 91.

One will determine the numbers $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ after the conditions

$$(2) = \vartheta_1(1) + \vartheta_2(5) + \vartheta_3(11) + \vartheta_4(15)$$

and

$$\vartheta_0\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 = 1;$$

it becomes

$$\vartheta_0 = 1,\ \vartheta_1 = -1,\ \vartheta_2 = 1,\ \vartheta_3 = 0,\ \vartheta_4 = 0.$$

It follows that one will replace the three pairs of simplexes

$$(0,1,5,11,15) \text{ and } (0,2,5,11,15),$$
$$(0,1,5,12,15) \text{ and } (0,2,5,12,15),$$
$$(0,1,5,11,4') \text{ and } (0,2,5,11,4')$$

by the simplexes

$$\begin{array}{c}(0,1,5,11,15) \text{ and } (0,1,2,11,15),\\ (2,1,5,12,15) \text{ and } (0,1,2,12,15),\\ (2,1,5,11,4') \text{ and } (0,1,2,11,4').\end{array} \qquad (3)$$

By designating the system $(1,-1,0,0)$ by the symbol (5) and the system $(-1,1,0,0)$ by the symbol $(5')$, one will designate

$$\begin{array}{rc} \text{I} - & (0,5,1,6,13),\\ \text{II} - & (0,1,2,11,15),\\ \text{V} - & (0,5,1,7,13),\\ \text{VI} - & (0,1,2,12,15),\\ \text{IX} - & (0,5,1,6,9'),\\ \text{X} - & (0,1,2,11,4'). \end{array}$$

These simplexes are congruent to the new simplexes (3).

The primitive parallelohedra of the IInd type possess 120 vertices which are brought together in the following table:

Vertices of the primitive parallelohedron of IInd type

I	1 5 6 13	1′ 2′ 3 10	2 5′ 8 14	3′ 8′ 6′ 4	10′ 14′ 4′ 13′
	1′ 5′ 6′ 13′	1 2 3′ 10′	2′ 5 8′ 14′	3 8 6 4′	10 14 4 13
II	1 2 11 15	1′ 5′ 8 14	5 2′ 6 13	8′ 6′ 11′ 4	14′ 13′ 4′ 15′
	1′ 2′ 11′ 15′	1 5 8′ 14′	5′ 2 6′ 13′	8 6 11 4′	14 13 4 15
III	1 6 11 4′	1′ 3 8 7′	3′ 6′ 2 13′	8′ 2′ 11′ 15′	7 13 15 4
	1′ 6′ 11′ 4	1 3′ 8′ 7	3 6 2′ 13	8 2 11 15	7′ 13′ 15′ 4′
IV	1 7 13 15	1′ 4 10 14	4′ 7′ 3 8	10′ 3′ 13′ 2	14′ 8′ 2′ 15′
	1′ 7′ 13′ 15′	1 4′ 10′ 14′	4 7 3′ 8′	10 3 13 2′	14 8 2 15
V	1 5 7 13	1′ 2′ 4 10	2 5′ 9 14	4′ 9′ 7′ 3	10′ 14′ 3′ 13′
	1′ 5′ 7′ 13′	1 2 4′ 10′	2′ 5 9′ 14′	4 9 7 3′	10 14 3 13
VI	1 2 12 15	1′ 5′ 9 14	5 2′ 7 13	9′ 7′ 12′ 3	14′ 13′ 3′ 15′
	1′ 2′ 12′ 15′	1 5 9′ 14′	5′ 2 7′ 13′	9 7 12 3′	14 13 3 15
VII	1 7 12 3′	1′ 4 9 6′	4′ 7′ 2 13′	9′ 2′ 12′ 15′	6 13 15 3
	1′ 7′ 12′ 3	1 4′ 9′ 6	4 7 2′ 13	9 2 12 15	6′ 13′ 15′ 3′
VIII	1 6 13 15	1′ 3 10 14	3′ 6′ 4 9	10′ 4′ 13′ 2	14′ 9′ 2′ 15′
	1′ 6′ 13′ 15′	1 3′ 10′ 14′	3 6 4′ 9′	10 4 13 2′	14 9 2 15
IX	1 5 6 9′	1′ 2′ 3 12′	2 5′ 8 7′	3′ 8′ 6′ 15′	12 7 15 9
	1′ 5′ 6′ 9	1 2 3′ 12	2′ 5 8′ 7	3 8 6 15	12′ 7′ 15′ 9′
X	1 2 11 4′	1′ 5′ 8 7′	5 2′ 6 9′	8′ 6′ 11′ 15′	7 9 15 4
	1′ 2′ 11′ 4	1 5 8′ 7	5′ 2 6′ 9	8 6 11 15	7′ 9′ 15′ 4′
XI	1 6 11 15	1′ 3 8 14	3′ 6′ 2 9	8′ 2′ 11′ 4	14′ 9′ 4′ 15′
	1′ 6′ 11′ 15′	1 3′ 8′ 14′	3 6 2′ 9′	8 2 11 4′	14 9 4 15
XII	1 7 12 15	1′ 4 9 14	4′ 7′ 2 8	9′ 2′ 12′ 3	14′ 8′ 3′ 15′
	1′ 7′ 12′ 15′	1 4′ 9′ 14′	4 7 2′ 8′	9 2 12 3′	14 8 3 15

Regulators and characteristic corresponding to the IInd-type of parallelohedra

I	0 1 5 6 13	–1 0 0 0 2 0 1 1	λ_1	1 1 –1 0 (2′) μ_1	0 –1 0 0 (15) λ_2	0 0 1 –1 (7) μ_6	0 0 0 1 (9′) $\lambda_4+\mu_1$
II	0 1 2 11 15	–1 –1 1 0 1 1 0 0	μ_1	1 0 –1 0 (8) μ_2	0 1 –1 0 (6) μ_4	0 0 1 –1 (12) $\mu_6+\mu_1$	0 0 0 1 (4′) λ_4
III	0 1 6 11 4′	–1 0 0 1 1 0 0 –1	$\mu_3+\mu_1$	1 0 –1 0 (8) μ_2	0 –1 1 0 (2) μ_4	0 1 0 0 (9′) $\lambda_2+\mu_1$	0 0 0 –1 (15) λ_4
IV	0 1 7 13 15	–1 0 0 0 2 0 1 1	λ_1	1 0 0 –1 (4) $\mu_3+\mu_1$	0 0 –1 1 (6) μ_6	0 –1 1 0 (12) $\mu_4+\mu_1$	0 1 0 0 (5) λ_2
V	0 1 5 7 13	–1 0 0 0 2 0 1 1	λ_1	1 1 0 –1 (2′) μ_1	0 –1 0 0 (15) λ_2	0 0 –1 1 (6) μ_6	0 0 1 0 (8′) $\lambda_3+\mu_1$
VI	0 1 2 12 15	–1 –1 0 1 1 1 0 0	μ_1	1 0 0 –1 (9) μ_3	0 1 0 –1 (7) μ_5	0 0 –1 1 (11) $\mu_6+\mu_1$	0 0 1 0 (3′) λ_3
VII	0 1 7 12 3′	–1 0 1 0 1 0 –1 0	$\mu_2+\mu_1$	1 0 0 –1 (9) μ_3	0 –1 0 1 (2) μ_5	0 1 0 0 (8′) $\lambda_2+\mu_1$	0 0 –1 0 (15) λ_3
VIII	0 1 6 13 15	–1 0 0 0 2 0 1 1	λ_1	1 0 –1 0 (3) $\mu_2+\mu_1$	0 0 1 –1 (7) μ_6	0 –1 0 1 (11) $\mu_5+\mu_1$	0 1 0 0 (5) λ_2
IX	0 1 5 6 9′	–1 0 0 1 1 –1 0 –1	μ_3	1 1 –1 –1 (2′) μ_1	0 –1 0 1 (4′) μ_5	0 0 1 0 (14′) λ_3	0 0 0 –1 (13) $\lambda_4+\mu_1$
X	0 1 2 11 4′	–1 –1 1 1 1 1 0 0	μ_1	1 0 –1 0 (8) μ_2	0 1 –1 0 (6) μ_4	0 0 1 0 (10′) $\lambda_3+\mu_1$	0 0 0 –1 (15) λ_4
XI	0 1 6 11 15	–1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 –1 0 (8) μ_2	0 –1 1 0 (2) μ_4	0 1 0 –1 (13) $\mu_5+\mu_1$	0 0 0 1 (4′) λ_4
XII	0 1 7 12 15	–1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 0 –1 (9) μ_3	0 –1 0 1 (2) μ_5	0 1 –1 0 (13) $\mu_4+\mu_1$	0 0 1 0 (3′) λ_3

The domain Δ' of quaternary quadratic forms which define the second type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Any quaternary quadratic form belonging to the domain Δ' can be written

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega \\ + \mu_2 (x_1 - x_3)^2 + \mu_3 (x_1 - x_4)^2 \\ + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_3 - x_4)^2 \end{aligned}$$

where

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4. \tag{4}$$

The parallelohedra belonging to the IInd type are determined by 30 inequalities of form (2) which are symbolically presented in the following table:

II^{nd}*-type of parallelohedra*

N	l_1	l_2	l_3	l_4	$f(l_1, l_2, l_3, l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	N
1	1	0	0	0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3$	−1	0	0	0	1′
2	0	1	0	0	$\lambda_2 + 2\mu 1_+ \mu_4 + \mu_5$	0	−1	0	0	2′
3	0	0	1	0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0	0	−1	0	3′
4	0	0	0	1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0	0	0	−1	4′
5	1	−1	0	0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	−1	1	0	0	5′
6	1	0	1	0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	−1	0	−1	0	6′
7	1	0	0	1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	−1	0	0	−1	7′
8	0	1	1	0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	0	−1	−1	0	8′
9	0	1	0	1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	0	−1	0	−1	9′
10	0	0	1	1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0	0	−1	−1	10′
11	1	1	1	0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	−1	−1	−1	0	11′
12	1	1	0	1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	−1	−1	0	−1	12′
13	1	0	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5$	−1	0	−1	−1	13′
14	0	1	1	1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3$	0	−1	−1	−1	14′
15	1	1	1	1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	−1	−1	−1	−1	15′

Let us find the domain Δ'' of quaternary quadratic forms which is contiguous to the domain Δ' by the face in 9 dimensions defined by the equation

$$\mu_6 = 0.$$

The independent regulator μ_6 corresponds to the faces of simplexes

I, *IV*, *V*, *VIII*.

All these simplexes have to be reconstructed after the algorithm explained in Number 91. One will determine, to this effect, the number

$$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$$

after the condition

$$\vartheta_1(1) + \vartheta_2(5) + \vartheta_3(6) + \vartheta_4(13) \quad \text{and} \quad \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4; \tag{5}$$

it becomes

$$\vartheta_0 = 0, \ \ \vartheta_1 = 1, \ \ \vartheta_2 = 0, \ \ \vartheta_3 = -1, \ \ \vartheta_4 = 1.$$

One concludes that the two pairs of simplexes

$$(0,1,5,6,13) \text{ and } (0,1,5,7,13) \text{ and}$$
$$(0,1,7,13,15) \text{ and } (0,1,6,13,15)$$

have to be replaced by the new simplexes

$$\text{I} - (0,7,5,6,13) \quad \text{and} \quad \text{V} - (0,1,5,6,7)$$
$$\text{IV} -(0,6,7,13,15) \quad \text{and} \quad \text{VIII} - (0,1,7,6,15).$$

By designating the system $(0,0,1,-1)$ by the symbol (10) and the system $(0,0,-1,1)$ by the symbol $(10')$ one will determine all the vertices of primitive parallelohedra belonging to the new type as follows.

Vertices of primitive parallelohedron of IIIrd-type.

I	7	5	6	13	7′	9′	10	3	9	5′	8	14	10′	8′	6′	4	3′	14′	4′	13′
	7′	5′	6′	13′	7	9	10′	3′	9′	5	8′	14′	10	8	6	4′	3	14	4	13
II	1	2	11	15	1′	5′	8	14	5	2′	6	13	8′	6′	11′	4	14′	13′	4′	15′
	1′	2′	11′	15′	1	5	8′	14′	5′	2	6′	13′	8	6	11	4′	14	13	4	15
III	1	6	11	4′	1′	3	8	7′	3′	6′	2	13′	8′	2′	11′	15′	7	13	15	4
	1′	6′	11′	4	1	3′	8′	7	3	6	2′	13	8	2	11	15	7′	13′	15′	4′
IV	6	7	13	15	6′	10′	4	9	10	7′	3	8	4′	3′	13′	2	9′	8′	2′	15′
	6′	7′	13′	15′	6	10	4′	9′	10′	7	3′	8′	4	3	13	2′	9	8	2	15
V	1	5	6	7	1′	2′	3	4	2	5′	8	9	3′	8′	6′	10′	4′	9′	10	7′
	1′	5′	6′	7′	1	2	3′	4′	2′	5	8′	9′	3	8	6	10	4	9	10′	7
VI	1	2	12	15	1′	5′	9	14	5	2′	7	13	9′	7′	12′	3	14′	13′	3′	15′
	1′	2′	12′	15′	1	5	9′	14′	5′	2	7′	13′	9	7	12	3′	14	13	3	15
VII	1	7	12	3′	1′	4	9	6′	1	4′	9′	6	4′	7′	2	13′	9′	2′	12′	15′
	6	13	15	3	1′	7′	12′	3	4	7	2	13	9	2	12	15	6′	13′	15′	3′
VIII	1	7	6	15	1′	4	3	14	4′	7′	10	8	3′	10′	6′	9	14′	8′	9′	15′
	1′	7′	6′	15′	1	4′	3′	14′	4	7	10′	8′	3	10	6	9′	14	8	9	15
IX	1	5	6	9′	1′	2′	3	12′	2	5′	8	7′	3′	8′	6′	15′	12	7	15	9
	1′	5′	6′	9	1	2	3′	12	2′	5	8′	7	3	8	6	15	12′	7′	15′	9′
X	1	2	11	4′	1′	5′	8	7′	5	2′	6	9′	8′	6′	11′	15′	7	9	15	4
	1′	2′	11′	4	1	5	8′	7	5′	2	6′	9	8	6	11	15	7′	9′	15′	4′
XI	1	6	11	15	1′	3	8	14	3′	6′	2	9	8′	2′	11′	4	14′	9′	4′	15′
	1′	6′	11′	15′	1	3′	8′	14′	3	6	2′	9′	8	2	11	4′	14	9	4	14
XII	1	7	12	15	1′	4	9	14	4′	7′	2	8	9′	2′	12′	3	14′	8′	3′	15′
	1′	7′	12′	15′	1	4′	9′	14′	4	7	2′	8′	9	2	12	3′	14	8	3	15

Regulators and characteristics corresponding to the IIIrd-type parallelohedra

I	0 7 5 6 13	−1 0 0 0 2 0 1 1	λ_1	1 1 −1 0 (2′) μ_1	0 −1 0 0 (15) λ_2	1 1 0 −1 (2′) μ_1	−1 −1 1 1 (1) μ_6
II	0 1 2 11 15	−1 −1 1 0 1 1 0 1	μ_1	1 0 −1 0 (8) μ_2	0 1 −1 0 (6) μ_4	0 0 1 −1 (12) μ_1	0 0 0 1 (4′) λ_4
III	0 1 6 11 4′	−1 0 0 1 1 0 1 −1	$\mu_3+\mu_1$	1 0 −1 0 (8) μ_2	0 −1 1 0 (2) μ_4	0 1 0 0 (9′) $\lambda_2+\mu_1$	0 0 0 −1 (15) λ_4
IV	0 6 7 13 15	−1 0 0 0 2 0 1 1	λ_1	1 0 0 −1 (4) $\mu_3+\mu_1$	1 0 −1 0 (3) $\mu_2+\mu_1$	−1 −1 1 1 (1) μ_6	0 1 0 0 (5) λ_2
V	0 1 5 6 7	−1 0 0 0 2 0 1 1	λ_1	1 1 −1 −1 (13) μ_6	0 −1 0 0 (15) λ_2	0 0 1 0 (8′) $\lambda_3+\mu_1$	0 0 0 1 (9′) $\lambda_4+\mu_1$
VI	0 1 2 12 15	−1 −1 0 1 1 1 1 0	μ_1	1 0 0 −1 (9) μ_3	1 1 0 −1 (7) μ_5	0 0 −1 1 (11) μ_1	0 0 1 0 (3′) λ_3
VII	0 1 7 12 3′	−1 0 0 0 1 0 −1 1	$\mu_2+\mu_1$	1 0 0 −1 (9) μ_3	0 −1 0 1 (2) μ_5	0 1 0 0 (8′) $\lambda_2+\mu_1$	0 0 −1 0 (15) λ_3
VIII	0 1 7 6 15	−1 0 0 0 2 0 1 1	λ_1	1 1 −1 −1 (13) μ_6	0 −1 0 1 (11) $\mu_5+\mu_1$	0 −1 1 0 (12) $\mu_4+\mu_1$	0 1 0 0 (5) λ_2
IX	0 1 5 6 9′	−1 0 0 1 1 −1 0 −1	μ_3	1 1 −1 −1 (2′) $\mu_1+\mu_6$	0 −1 0 1 (4′) μ_5	0 0 1 0 (14′) λ_3	0 0 0 −1 (7) $\lambda_4+\mu_1$
X	0 1 2 11 4′	−1 −1 1 1 1 1 0 0	$\mu_1+\mu_6$	1 0 −1 0 (8) μ_2	0 1 −1 0 (6) μ_4	0 0 1 0 (3′) $\lambda_3+\mu_1$	0 0 0 −1 (15) λ_4
XI	0 1 6 11 15	−1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 −1 0 (8) μ_2	0 −1 1 0 (2) μ_4	0 1 0 −1 (7) $\mu_5+\mu_1$	0 0 0 1 (4′) λ_4
XII	0 1 7 12 15	−1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 0 −1 (9) μ_3	0 −1 0 1 (2) μ_5	0 1 −1 0 (6) $\mu_4+\mu_1$	0 0 1 0 (3′) λ_3

The independent regulators are expressed by the formulae

$$\begin{gathered}
\lambda_1 = a_{11} + a_{13} + a_{14} + a_{34}, \\
\lambda_2 = a_{22} + a_{23} + a_{24} + a_{34}, \\
\lambda_3 = a_{31} + a_{32} + a_{33} + a_{34}, \\
\lambda_4 = a_{41} + a_{42} + a_{43} + a_{44}, \\
\mu_1 = a_{12} - a_{34}, \quad \mu_2 = -a_{13} - a_{12}, \quad \mu_3 = -a_{14} - a_{12}, \\
\mu_4 = -a_{23} - a_{12}, \quad \mu_5 = -a_{24} - a_{12}, \quad \mu_6 = a_{34}.
\end{gathered}$$

The domain Δ'' of quaternary quadratic forms which defines the third type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Each quaternary quadratic form belonging to the domain Δ'' can be written

$$\begin{aligned}
f(x_1, x_2, x_3, x_4) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega \\
& + \mu_2 (x_1 - x_3)^2 + \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 \\
& + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_1 + x_2 - x_3 - x_4)^2,
\end{aligned}$$

the form ω being defined by the equality (4). The parallelohedra belonging to the IIIrd type are determined by 30 inequalities of the form (2) which are symbolically presented in the following table:

IIIrd type of parallelohedra

N	$l_1\ l_2 l_3 l_4$	$f(l_1,l_2,l_3,l_4)$	$-l_1 -l_2 -l_3 -l_4$	N
1	1 0 0 0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	−1 0 0 0	1′
2	0 1 0 0	$\lambda_2 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	0 −1 0 0	2′
3	0 0 1 0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0 0 −1 0	3′
4	0 0 0 1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0 0 0 −1	4′
5	1 −1 0 0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	−1 1 0 0	5′
6	1 0 1 0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4$	−1 0 −1 0	6′
7	1 0 0 1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5$	−1 0 0 −1	7′
8	0 1 1 0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5$	0 −1 −1 0	8′
9	0 1 0 1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4$	0 −1 0 −1	9′
10	0 0 1 −1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0 0 −1 1	10′
11	1 1 1 0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	−1 −1 −1 0	11′
12	1 1 0 1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	−1 −1 0 −1	12′
13	1 0 1 1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	−1 0 −1 −1	13′
14	0 1 1 1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	0 −1 −1 −1	14′
15	1 1 1 1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	−1 −1 −1 −1	15′

We have determined three domains $\Delta, \Delta', \Delta''$ which characterise three types of primitive parallelohedra in four dimensions.

Theorem. *The set* (Δ) *of domains of quaternary quadratic forms is composed of three different classes which can be represented by the domain* $\Delta, \Delta', \Delta''$.

In my first mémoire cited, it has been demonstrated that the set (R) of domains of quaternary quadratic forms corresponding to the perfect quaternary quadratic forms is composed of two classes represented by the principal domain R and by a domain R, determined by the equalities

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 \\ +\rho_5 (x_1 - x_3)^2 + \rho_6 (x_1 - x_4)^2 + \rho_7 (x_1 - x_3)^2 \\ +\rho_8 (x_2 - x_4)^2 + \rho_9 (x_3 - x_4)^2 + \rho_{10} (x_1 + x_2 - x_3)^2 \\ +\rho_{11} (x_1 + x_2 - x_4)^2 + \rho_{12} (x_1 + x_2 - x_3 - x_4)^2 \end{aligned}$$

where $\rho_1, \rho_2, \ldots, \rho_{12}$ are positive arbitrary parameters or zeros.

The domain R, corresponds to a perfect form

$$\varphi_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4. \quad (5)$$

In the mémoire cited, it has been demonstrated that all the faces in 9 dimensions of domain R, are equivalent to two faces characterised: one by the quadratic form

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2$$

and the other by the quadratic form

$$\begin{aligned} x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, \\ (x_1 + x_2 - x_3 - x_4)^2. \end{aligned}$$

The first face verifies the equation

$$a_{12} = 0$$

and the second face verifies the equation

$$a_{12} - a_{34} = 0$$

The form ω determined by the formula (4) characterise the axis of the domain (R) which does not change when one transforms the domain R into itself.

One concludes that the domain R can be partitioned into groups all of which are equivalent to the two domains Δ' and Δ'' obtained. This results in that the principal domain Δ and the two domains Δ' and Δ'' can not be equivalent.

The theorem introduced is thus demonstrated.

By not considering as different the equivalent types of parallelohedra one can say that there are only three different types of primitive parallelohedra in the space in four dimensions.

By calling reduced the positive quadratic forms which belong to the domains Δ, Δ' and Δ'' one obtains a new method of reduction of quaternary positive quadratic forms which presents a modification of the method due to Mr. Charve. ‡

In effect, following the method of Mr. Charve, one calls reduced the quaternary positive quadratic forms belonging to one of three simple domains R, R' and R''. The first two domain R and R' coincide with the domains Δ and Δ' and it is only the third domain R'' of Mr. Charve which differs from the domain Δ''. Any form belonging to the domain Δ'' is equivalent to a form belonging to the domain R'' and vice-versa.

By examining the two tables which contain the characteristics of faces of simplexes which define the 2^{nd} and the 3^{rd} type of primitive parallelohedra, one will observe that these characteristic coincide for the two types and are represented by the linear forms

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‡ See the introduction

$$\begin{gathered}\pm x_1, \pm x_2, \pm x_3, \pm x_4,\\ \pm(x_1 - x_3), \pm(x_1 - x_4), \pm(x_2 - x_3), \pm(x_2 - x_4), \pm(x_3 - x_4),\\ \pm(x_1 + x_2 - x_3), \pm(x_1 + x_2 - x_4),\\ \pm(x_1 + x_2 - x_3 - x_4).\end{gathered}$$

It is remarkable that these linear forms define the set of representations of the minimum of the perfect form φ_1 determined by the equality (5).

By virtue of that which has previously been mentioned, one can affirm that the coincidence noticed appears as the characteristics of faces of all the primitive parallelohedra in $2, 3$ and 4 dimensions.

It would be interesting to find out whether this is only a coincidence or whether there really exists a relation between the two problems which seem to be different: between the problem of the uniform partition of the space with the help of congruent convex polyhedra and the study of perfect positive quadratic forms.

End of the second mémoire

[in German]

Immediately after the first sheet of this significant work was set, we received the grievous tidings that your author of the science has been taken away by Death. The editor had in the best power seen to it that this last work of he who so early departed for the other side was checked over with utmost care.

Marburg, 19th June 1909

K. Hensel

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